The horizontal range of the projectile motion:

An object is launched from a ground and is returned to its original height. The time for the entire travel of the projectile motion is given by the y direction of the motion.

$$y = y_0 + v_{y0}t - \frac{1}{2}gt^2 \tag{1}$$

The initial and final height can be the reference position, zero. Namely,

$$0 = 0 + v_{y0}t - \frac{1}{2}gt^2 \tag{2}$$

The quadratic equation gives two solutions that are $t = 0$ and $t = \frac{2v_{y0}}{g}$. The latter solution expresses the entire time traveled.

The horizontal range is given by

$$x = v_{x0}t \tag{3}$$

Plug in the time obtained the above.

$$x = v_{x0}\frac{2v_{y0}}{g} = \frac{2v_0^2 \cos\theta \sin\theta}{g} \tag{4}$$

Using a formula of trigonometry, $2\sin\theta\cos\theta = \sin 2\theta$, we have

$$x = \frac{v_0^2 \sin 2\theta}{g} \tag{5}$$

The height of the projectile motion:

Let the initial position be origin. The x- and y-components of a projectile motion are given as follows:

$$x = v_{x0}t$$
$$y = v_{y0}t - \frac{1}{2}gt^2 \tag{6}$$

The time can be solved as

$$t = \frac{x}{v_{x0}} \tag{7}$$

This is plugged in to the y component of the equation.

$$y = \left(\frac{v_{y0}}{v_{x0}}\right)x - \left(\frac{g}{2v_{x0}^2}\right)x^2 \tag{8}$$

The form of the equation is known as a parabola. Use $v_{x0} = v_0 \cos\theta$ and $v_{y0} = v_0 \sin\theta$ to modify the above expression.

$$y = \tan\theta \cdot x - \left(\frac{g}{2v_0^2 \cos^2\theta}\right)x^2 \tag{9}$$

To obtain the maximum height, take the derivative of (9).

$$y' = \tan\theta - 2 \cdot \left(\frac{g}{2v_0^2 \cos^2\theta}\right)x \tag{10}$$

The following position gives the maximum value of *y*.

$$x = \frac{\tan\theta}{2 \cdot \left(\frac{g}{2v_0^2 \cos^2\theta}\right)} \tag{11}$$

Let $g/\left(2v_0^2 \cos^2\theta\right)$ be G. Plug (11) into (9).

$$\begin{aligned} y &= \tan\theta \frac{\tan\theta}{2G} - G\frac{\tan^2\theta}{4G^2} \\ &= \frac{2\tan^2\theta - \tan^2\theta}{4G} \\ &= \frac{\tan^2\theta}{4G} \\ &= \frac{\tan^2\theta}{\frac{2g}{v_0^2 \cos^2\theta}} \\ &= \frac{v_0^2 \cos^2\theta \tan^2\theta}{2g} \\ &= \frac{v_0^2 \sin^2\theta}{2g} \end{aligned} \tag{12}$$

This is the maximum height of a projectile motion.