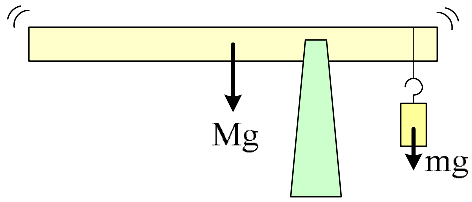
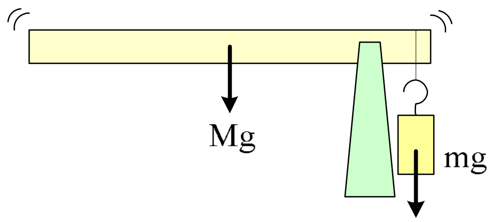
Why does the gravitational force exert the center of mass?

In the lab, Equilibrium of Torques, you make a balance of torques like this:

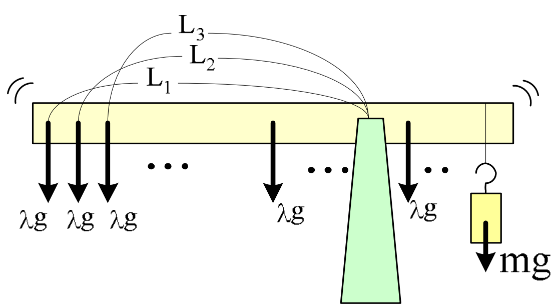


The right hand side of the torque will be the gravitational force of the weight (mg) × lever arm (from the fulcrum to the hanging mass). However, someone could think why the torque of the stick is calculated as the gravitational force due to the mass of stick (M) × the distance from "fulcrum" to the center of mass of stick. Let's see following:

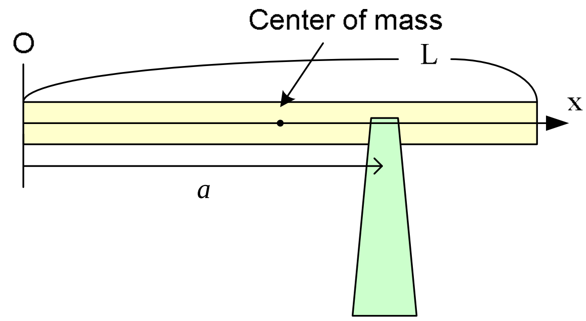


When you change the position of fulcrum, you have to use more weight for the hanging mass. However, the gravitational force of stick exerts the same point (the center of mass). Is it coincidence?

In fact, to obtain the total torque of the meter stick, you have to sum up all points of torques for the stick.



Namely, $\tau = L_1\lambda g + L_2\lambda g + L_3\lambda g + \ldots$ where λ is the linear density (density per unit length) of the stick. Of course, the space between L_1 and L_2 (L_2 and L_3 also) is very small. In this case, you can use integral to calculate it. Prepare the following coordinate.



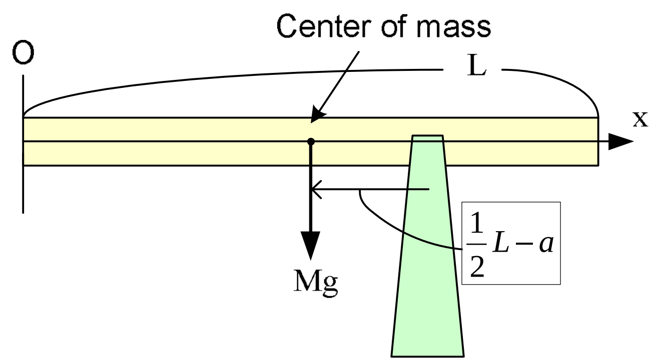
The origin is the left edge of the stick. The distance from the origin to the fulcrum is a. The length of the stick is L. If you consider the directions of torque at some point, you recognize: $x > a$ means that the point where a force exerts is right side of the fulcrum; and $x < a$ means that the point is placed on the left side.

$$\begin{cases} x > a & \tau > 0 \\ x < a & \tau < 0 \end{cases}$$

Using the above conditions, you will sum up each point of torque; namely, set up the following integral.

$$\tau = \int_0^L (x-a) g\lambda\, dx = \tfrac{1}{2} g\lambda L^2 - ag\lambda L$$
$$= Mg\left(\tfrac{1}{2}L - a\right) \qquad \because M = \lambda L$$

Mg is the gravitational force, and this vector, $\left(\tfrac{1}{2}L - a\right)$, expresses:



If you assume that the density of stick is uniform, $\tfrac{1}{2}L$ is the center of mass, and a is arbitrary. Therefore, the gravitational force due to the stick always exerts the center of gravity.

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