Mathematical Aspect of Fourier Series

Historically, there was an assumption that any function can be expressed by a series of trigonometric functions. A mathematician, Fourier, derived the formula so that we can obtain such series. The general expression of the formula can be shown as follows:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \qquad \text{(Fourier Series of } f(x)\text{)}$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$, and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$. $f(x)$ is a given function you want to approximate with the series, a_n and b_n are called Fourier coefficients. We assume that $2L = \lambda$ (wave length).

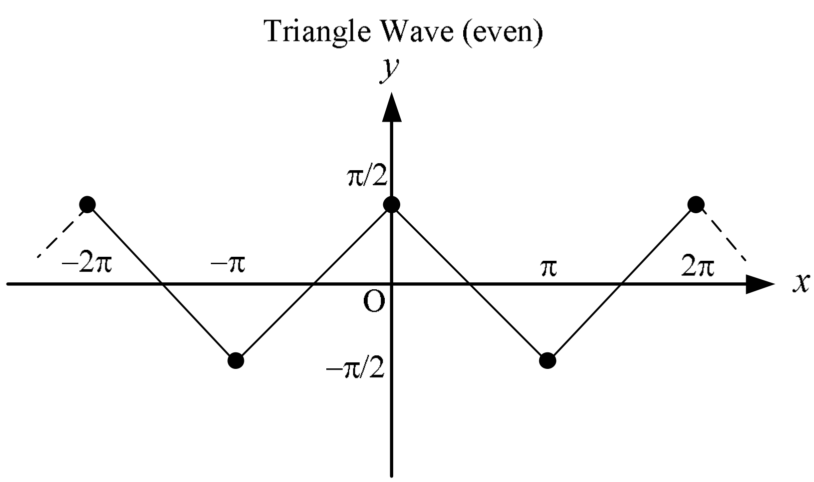
However, if you know the function as odd or even, you can use only sine or cosine for the series. Let us explain an even or odd function? Simply put, when you fold the graph paper with respect to *y*-axis, then the both sides of plots are identically overlapped. This type of function is even. For the odd one, it is symmetric with respect to origin. You can see cos(*x*) is an even function, and sin(*x*) is an odd function. Now, considering even and odd functions, we have the following simplified expressions for Fourier series.

For an even function: $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$ where $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$.

For an odd function: $f(x) \approx \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ where $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$.

Here are two example problems. The function is called the periodical triangle wave. First, we have the even one:

$$f(x) = \begin{cases} \frac{\pi}{2} + x \ (-\pi \le x \le 0) \\ \frac{\pi}{2} - x \ (0 \le x \le \pi) \end{cases}$$



From the formula and the graph, you find $a_0 = 0$, and $L = \pi$. Then we can calculate a_n as follows:

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} + x\right)\cos nx\,dx$$
$$= \frac{2}{\pi}\left[\int_0^{\pi}\frac{\pi}{2}\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2n}\sin nx\right)_0^{\pi} + \left(\frac{x}{n}\sin nx\right)_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right]$$
$$= \frac{2}{\pi}\left[-\frac{1}{n^2}\left(\cos nx\right)_0^{\pi}\right]$$
$$= \frac{2(1-\cos n\pi)}{n^2\pi} = \frac{2(1-(-1)^n)}{n^2\pi} \quad (n = 1,2,....).$$

However, if n's are even numbers, then $a_n = 0$. Therefore, the series will become

$$f(x) \approx \sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n^2\pi}\cos nx \quad \text{(odd } n \text{ only)},$$

or we can rewrite it as $f(x) \approx \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}$, since $n = 2k-1 \quad (k = 1,2,3...)$.

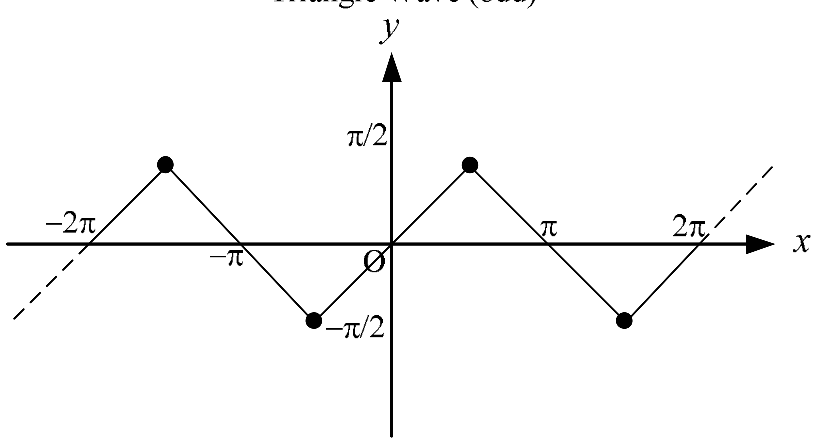
If you express it in term by term, you will have

$$f(x) \approx \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots\right).$$

Next you have the odd one, which is:

$$f(x) = \begin{cases} -\pi - x & (-\pi \le x \le -\frac{\pi}{2}) \\ x & (-\frac{\pi}{2} \le x \le \frac{\pi}{2}) \\ \pi - x & (\frac{\pi}{2} \le x \le \pi) \end{cases} \qquad (L = \pi)$$

Triangle Wave (odd)



Then use the second formula:

$$b_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin nx\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin nx\,dx\right]$$
$$= \frac{2}{\pi}\left[\left(-\frac{1}{n}x\cos nx\right)_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}}\cos nx\,dx + \left(-\frac{\pi}{n}\cos nx\right)_{\frac{\pi}{2}}^{\pi} + \left(\frac{1}{n}x\cos nx\right)_{\frac{\pi}{2}}^{\pi} - \frac{1}{n}\int_{\frac{\pi}{2}}^{\pi}\cos nx\,dx\right]$$

For odd n $\rightarrow$ $= \frac{2}{\pi}\left[0 + \frac{1}{n^2}\left(\sin nx\right)_0^{\frac{\pi}{2}} + \frac{\pi}{n} - \frac{\pi}{n} - \frac{1}{n^2}\left(\sin nx\right)_{\frac{\pi}{2}}^{\pi}\right]$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}(-1)^{l+1} + \frac{1}{n^2}(-1)^{l+1}\right] \quad (l = 1,2,3...)$$
$$= \frac{1}{\pi}\frac{4}{n^2}(-1)^{l+1}$$

For even n $\rightarrow$ $= \frac{2}{\pi}\left[(-1)^{l+1}\frac{\pi}{2n} + 0 - \frac{\pi}{n} - (-1)^{l+1}\frac{\pi}{n} + \frac{\pi}{n} + (-1)^{l+1}\frac{\pi}{2n} + 0\right] = 0.$

Therefore, we obtain the Fourier series of this function as

$$f(x) \approx \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin(2k-1)x,$$

which will be explicitly expressed as

$$f(x) \approx \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2}\cdots\right].$$

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What is usefulness of Fourier series? Sine and cosine are easily differentiated and integrated, so if you express some periodic function in terms of sine and cosine, it will be very convenient to calculate and analyze it.