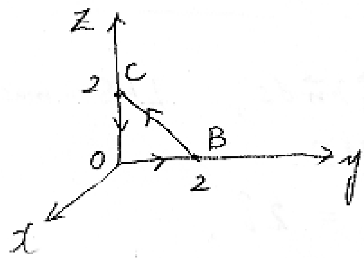
The Stokes Theorem & the Contour Integral

Example 1.

A vector is given as

$$\vec{A} = x\hat{i} + y\hat{j} + z\hat{k}.$$

Calculate $\oint_c \vec{A}\cdot d\vec{r}$ with the following contour.



* Note

$\hat{e}_x \equiv \hat{i} \equiv \hat{x}$

$\hat{e}_y \equiv \hat{j} \equiv \hat{y}$

$\hat{e}_z \equiv \hat{k} \equiv \hat{z}$

Solution

$$\oint_c \vec{A}\cdot d\vec{r} = \oint_c (x\hat{i}+y\hat{j}+z\hat{k})(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \int_{OB} (\cancel{x\hat{i}}+y\hat{j}+\cancel{z\hat{k}})(\cancel{dx\hat{i}}+dy\hat{j}+\cancel{dz\hat{k}})$$

$$+ \int_{BC} (\cancel{x\hat{i}}+y\hat{j}+z\hat{k})(\cancel{dx\hat{i}}+dy\hat{j}+dz\hat{k})$$

$$+ \int_{CO} (\cancel{x\hat{i}}+\cancel{y\hat{j}}+z\hat{k})(\cancel{dx\hat{i}}+\cancel{dy\hat{j}}+dz\hat{k})$$

$$= \int_{OB} y\hat{j}\,dy\hat{j} + \int_{BC}(y\hat{j}+z\hat{k})(dy\hat{j}+dz\hat{k}) + \int_{CO} z\hat{k}\,dz\hat{k}$$

* Note

$\hat{i}\cdot\hat{i} = 1$

$\hat{i}\cdot\hat{j} = 0$

$\vdots$

However, for BC, it is expressed $z = -y+2$, so $dz = -dy$.

$$\oint_c \vec{A}\cdot d\vec{r} = \int_{OB} y\,dy + \int_{BC}(y\,dy + z\,dz) + \int_{CO} z\,dz$$

$$= \int_0^2 y\,dy + \int_2^0 \{y\,dy - (-y+2)dy\} + \int_2^0 z\,dz$$

↖ y changes from 2 to zero.

$$= \left[\frac{y^2}{2}\right]_0^2 + \left[y^2 - 2y\right]_2^0 + \left[\frac{z^2}{2}\right]_2^0$$

$$= 2 + 0 - 2 = 0$$

* Note

From $\oint_c \vec{A}\cdot d\vec{r} = \int_S (\vec{\nabla}\times\vec{A})\cdot\vec{n}\,dS$,

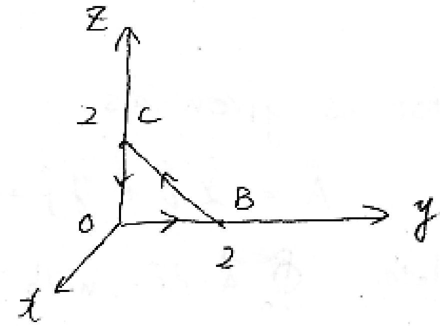
$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ x & y & z \end{vmatrix} = 0$$

Therefore, $\int_S (\vec{\nabla}\times\vec{A})\cdot\vec{n}\,dS = 0$.

And $\oint_c \vec{A}\cdot d\vec{r} = 0$.

Example 2. Repeat the same calculation with

$$\vec{A} = x\hat{i} - z\hat{j} + y\hat{k}$$



Solution

① From $\oint_c \vec{A}\cdot d\vec{r} = \int_S (\vec{\nabla}\times\vec{A})\cdot\vec{n}\,ds$, LHS must be equal to RHS. Calculate RHS.

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ x & -z & y \end{vmatrix} = 2\hat{i},$$

and $\vec{n}$ (normal vector) is $\hat{i}$, so

$$\int_S (\vec{\nabla}\times\vec{A})\cdot\vec{n}\,ds = \int_S 2\hat{i}\cdot\hat{i}\,ds = 2\int_S dS$$

$$= 2\int_0^2 (-y+2)dy = 4$$

② $$\oint_c \vec{A}\cdot d\vec{r} = \int_{OB} -z\hat{j}\,dy\hat{j} + \int_{BC}(-z\hat{j}\,dy\hat{j} + y\hat{k}\,dz\hat{k}) + \int_{CO} y\hat{k}\,dz\hat{k}$$

$$= \left[-zy\right]_0^2 + \int_2^0 \{(y-2)\,dy - y\,dy\} + \left[yz\right]_2^0$$

because
$$\begin{cases} z = -y+2 \\ dz = -dy \end{cases}$$

$$= -2z + 4 - 2y$$

However, for $\int_{OB} dy$, z is always zero, and for $\int_{CO} dz$, y is always zero. Therefore,

$$\oint_c \vec{A}\cdot d\vec{r} = 4.$$

This matches with the result in ①.