Problem:

Find the cosine transformation of $\cos x$:

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty}\cos x \cos k \; dx \;\; = ?$$

Solution:

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty}\cos x \cos k \; dx$$

Use the addition theorem of trigonometric function.

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty}\{\cos(1+k)x+\cos(1-k)x\}dx$$

$$=\sqrt{\frac{1}{2\pi}}\left[\frac{\sin(1+k)x}{1+k}+\frac{\sin(1-k)x}{1-k}\right]_0^{\infty}$$

When substituting ∞ in sine, the answer will be indeterminate. Put it for α to have a meaningful solution.

$$=\sqrt{\frac{1}{2\pi}}\left[\frac{1}{1-k^2}\{(1-k)\sin(1+k)x+(1+k)\sin(1-k)x\}\right]_0^{\alpha}$$

$$=\sqrt{\frac{1}{2\pi}}\frac{1}{1-k^2}((1-k)\sin(1+k)\alpha+(1+k)\sin(1-k)\alpha)$$

$$=\sqrt{\frac{1}{2\pi}}\frac{1}{1-k^2}((1-k)\{\sin\alpha\cos\alpha k+\sin\alpha k\cos\alpha\}+(1+k)\{\sin\alpha\cos\alpha k-\sin\alpha k\cos\alpha\})$$

$$=\sqrt{\frac{1}{2\pi}}\frac{1}{1-k^2}(\sin\alpha\cos\alpha k-k\sin\alpha\cos\alpha k+\sin\alpha k\cos\alpha-k\sin\alpha k\cos\alpha$$
$$+\sin\alpha\cos\alpha k+k\sin\alpha\cos\alpha k-\sin\alpha k\cos\alpha-k\sin\alpha k\cos\alpha)$$

$$=\sqrt{\frac{1}{2\pi}}\frac{1}{1-k^2}(2\sin\alpha\cos\alpha k-2k\sin\alpha k\cos\alpha)$$