Quaternion

A quaternion number is expressed as

$$Q = a + bi + cj + dk \tag{1}$$

The, i, j, k are defined as follows:

$$i^2 = j^2 = k^2 = -1$$

and

$$ij = k, \quad jk = i, \quad ki = j$$

$$ji = -k, \quad kj = -i, \quad ik = -j$$

The product table shows

	1	***i***	***j***	***k***
1	***1***	***i***	***j***	***k***
i	***i***	***-1***	***k***	***-j***
j	***j***	***-k***	***-1***	***i***
k	***k***	***j***	***-i***	***-1***

Therefore,

$$ijk = -1, \quad ikj = 1, \quad jik = 1, \quad jki = -1, \quad kji = 1, \quad kij = -1$$

The quaternion conjugate of (1) is

$$\overline{Q} = a - bi - cj - dk$$

The norm becomes

$$\sqrt{Q\overline{Q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

For the sum and product, they are calculated as follows. Define

$Q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $Q_2 = a_2 + b_2 i + c_2 j + d_2 k$

$$Q_1 + Q_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k;$$

$$\begin{aligned} Q_1 Q_2 &= (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) + (a_1 b_2 + b_1 a_2 + c_1 d_2 + d_1 b_2)i \\ &\quad + (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2)j + (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2)k \end{aligned}$$

Split- Quaternion or Coquaternion

The product table shows

	1	*i*	*j*	*k*
1	*1*	*i*	*j*	*k*
i	*i*	*-1*	*k*	*-j*
j	*j*	*-k*	*1*	*-i*
k	*k*	*j*	*i*	*1*

Octonion

Here is a table of all combinations of octonions:

	1	*i*	*j*	*k*	*l*	*il*	*jl*	*kl*
1	*1*	*i*	*j*	*k*	*l*	*il*	*jl*	*kl*
i	*i*	*-1*	*k*	*-j*	*il*	*-l*	*-kl*	*jl*
j	*j*	*-k*	*-1*	*i*	*jl*	*kl*	*-l*	*-il*
k	*k*	*j*	*-i*	*-1*	*kl*	*-jl*	*il*	*-l*
l	*l*	*-il*	*-jl*	*-kl*	*-1*	*i*	*j*	*k*
il	*il*	*l*	*-kl*	*jl*	*-i*	*-1*	*-k*	*j*
jl	*jl*	*kl*	*l*	*-il*	*-j*	*k*	*-1*	*-i*
kl	*kl*	*-jl*	*il*	*l*	*-k*	*-j*	*i*	*-1*

Octonions do not hold the commutative law, and they do not hold the associative law either. Therefore, $(ij)l \neq i(jl) = -i(jl)$.

The other expression of the product is following:

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Octonions hold the following relationship with norms:

$$\|ab\| = \|a\|\|b\|$$