Zeta function

The simple definition of Zeta function is

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

Under this definition, $s=1$ or $s<1$ gives infinity. When $s>1$, the function will be converged into some value. For example,

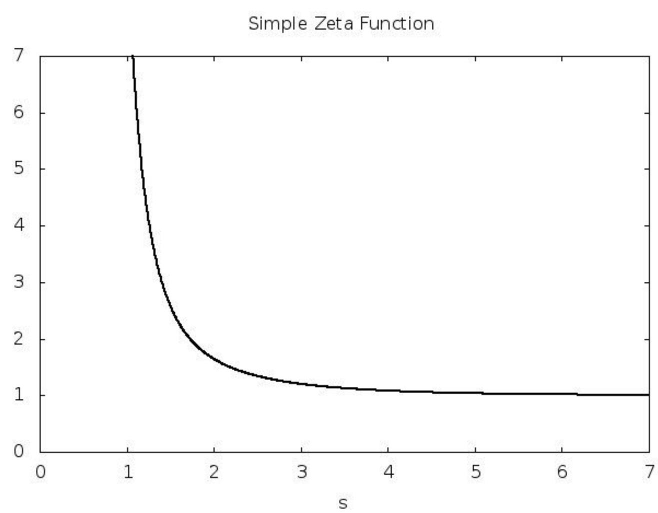
$$\zeta(2)=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{6}$$

$$\zeta(4)=1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\cdots=\frac{\pi^4}{90}$$

$$\zeta(6)=1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\cdots=\frac{\pi^6}{945}$$

$$\zeta(12)=1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\cdots=\frac{691\pi^{12}}{638512875}$$

The converged values in terms of *s* are plotted as shown below:



The completed Riemann-Zeta function is shown as follows:

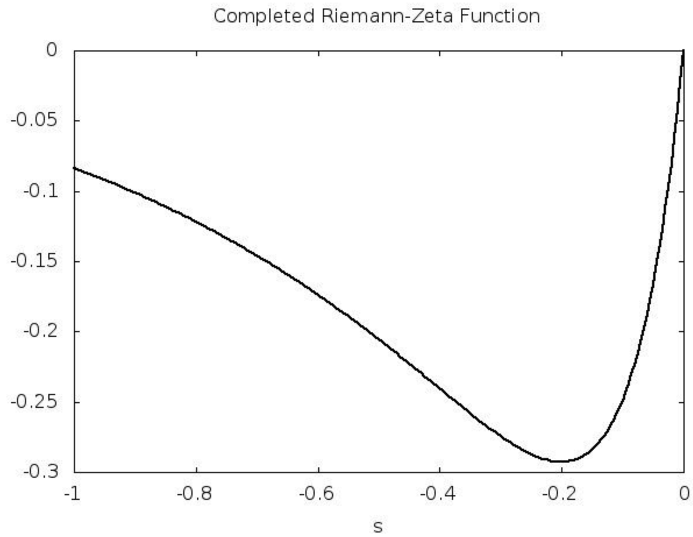
$$\zeta_R(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

or

$$\zeta_R(s):=\frac{1}{\sqrt{\pi^s}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$= \int_1^\infty \left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}} \right) \left(\sum_{n=1}^{\infty} e^{-\pi n^2 x} \frac{dx}{x} \right) + \frac{1}{s(1-s)}$$

The other expressions are also found in any references. The plot of this function is shown below:



This focuses on the range, $-1 \le s \le 0$. When $s = -1$, the function has the value, -0.08333, which is -1/12. Therefore, we can have

$$\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots = -\frac{1}{12}$$