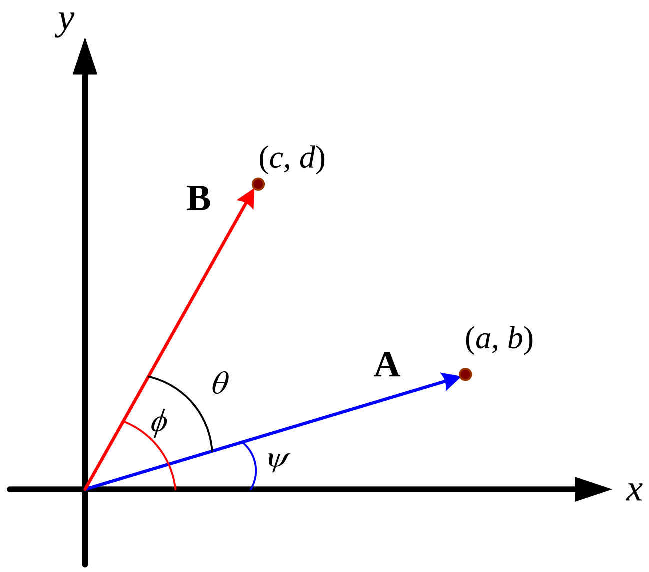
How to derive the algebraic inner product from the geometric one

The algebraic inner product is calculated by multiplying the same components and adding them up. For example, suppose we have following two vectors:



$$\vec{A}=(a,b) \quad \vec{B}=(c,d)$$

The result of the inner product is:

$$\vec{A}\cdot\vec{B}=ac+bd \qquad (1)$$

On the other hand, the geometric expression is:

$$\vec{A}\cdot\vec{B}=\left|\vec{A}\right|\left|\vec{B}\right|\cos\theta \qquad (2)$$

The θ is the angle between vectors A and B as shown in the figure.

Now, we find each angle, and its cosine is:

$$\cos\phi=\frac{c}{\sqrt{c^2+d^2}} \quad \cos\psi=\frac{a}{\sqrt{a^2+b^2}} \qquad (3)$$

Therefore,

$$\phi=\cos^{-1}\frac{c}{\sqrt{c^2+d^2}} \quad \psi=\cos^{-1}\frac{a}{\sqrt{a^2+b^2}} \qquad (4)$$

Since $\theta=\phi-\psi$, we have

$$\theta=\cos^{-1}\frac{c}{\sqrt{c^2+d^2}}-\cos^{-1}\frac{a}{\sqrt{a^2+b^2}} \qquad (5)$$

The formula of the sum of arccosine is given as

$$\cos^{-1}\alpha\pm\cos^{-1}\beta=\cos^{-1}\left(\alpha\beta\mp\sqrt{\left(1-\alpha^2\right)\left(1-\beta^2\right)}\right) \qquad (6)$$

Thus, (5) becomes

$$\theta=\cos^{-1}\left[\frac{ca}{\sqrt{c^2+d^2}\sqrt{a^2+b^2}}+\sqrt{\left(1-\frac{c^2}{c^2+d^2}\right)\left(1-\frac{a^2}{a^2+b^2}\right)}\right] \qquad (7)$$

We also know

$$\left|\vec{A}\right| = \sqrt{a^2+b^2} \quad \left|\vec{B}\right| = \sqrt{c^2+d^2} \tag{8}$$

Then, plug (7) and (8) into (2).

$$\vec{A}\cdot\vec{B} = \sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos\left[\cos^{-1}\left[\frac{ca}{\sqrt{c^2+d^2}\sqrt{a^2+b^2}} + \sqrt{\left(1-\frac{c^2}{c^2+d^2}\right)\left(1-\frac{a^2}{a^2+b^2}\right)}\right]\right]$$

$$= \sqrt{a^2+b^2}\sqrt{c^2+d^2}\left[\frac{ca}{\sqrt{c^2+d^2}\sqrt{a^2+b^2}} + \sqrt{\left(1-\frac{c^2}{c^2+d^2}\right)\left(1-\frac{a^2}{a^2+b^2}\right)}\right]$$

$$= ca + \sqrt{c^2+d^2}\sqrt{a^2+b^2}\sqrt{\left(1-\frac{c^2}{c^2+d^2}\right)\left(1-\frac{a^2}{a^2+b^2}\right)}$$

$$= ca + \sqrt{\left(c^2+d^2\right)\left(a^2+b^2\right)\left(1-\frac{c^2}{c^2+d^2}\right)\left(1-\frac{a^2}{a^2+b^2}\right)}$$

$$= ca + \sqrt{\left(c^2+d^2\right)\left(1-\frac{c^2}{c^2+d^2}\right)\left(a^2+b^2\right)\left(1-\frac{a^2}{a^2+b^2}\right)}$$

$$= ca + \sqrt{\left(c^2+d^2-c^2\right)\left(a^2+b^2-a^2\right)}$$

$$= ca + \sqrt{b^2d^2}$$

$$= ca + bd$$

This result is equal to that of (1).

Appendix

Derive the sum of arccosine:

We put

$$\alpha = \cos A \quad \beta = \cos B \tag{a}$$

Thus,

$$A = \cos^{-1}\alpha \quad B = \cos^{-1}\beta \tag{b}$$

Sum of inside cosine is given as:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad \text{(c)}$$

Take arccosine of both hands.

$$A \pm B = \cos^{-1}\left[\cos A \cos B \mp \sin A \sin B\right] \qquad \text{(d)}$$

The sinA and sinB are expressed as

$$\sin A = \sqrt{1-\cos^2 A} = \sqrt{1-\alpha^2} \quad \sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\beta^2} \qquad \text{(e)}$$

Use (b) and (e) to plug in (d).

$$\cos^{-1}\alpha \pm \cos^{-1}\beta = \cos^{-1}\left[\alpha\beta \mp \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\right]$$