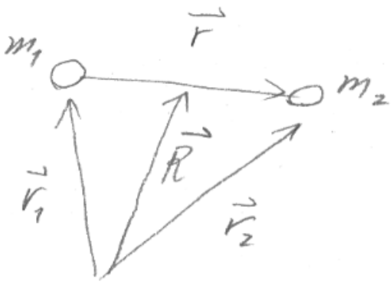
The kinetic energy :



$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$$

where $\vec{R} := \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \left(\begin{array}{l} M = m_1 + m_2 \\ \mu := \dfrac{m_1 m_2}{M} \end{array} \right)$$

The Lagrangian will be

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - V(r)$$

Since the conjugate momenta are :

$$P_X = \frac{\partial L}{\partial \dot{X}} = M\dot{X} \quad \text{------ etc.}$$

$$P_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r} \quad \text{------ etc.}$$

The Hamiltonian will be

$$H = \frac{1}{2M}(P_X^2 + P_Y^2 + P_Z^2) + \frac{1}{2\mu}\left(P_r^2 + \frac{P_\theta^2}{r^2} + \frac{P_\phi^2}{r^2\sin^2\theta}\right) + V(r)$$

X, Y, Z, and ϕ are cyclic.

The equations of motion for X, Y, and Z are :

$$\dot{X} = \frac{\partial H}{\partial P_X} = \frac{P_X}{M}, \quad \text{... etc.}$$

Thus,

$$X = \frac{P_X}{M}t + X_0, \quad \text{... etc.}$$

The other equations :

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu} \qquad \text{— (1)}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{1}{\mu r^3}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - \frac{dV}{dr} \qquad \text{— (2)}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mu r^2} \qquad \text{— (3)}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{\mu r^3 \sin^3\theta} \qquad \text{— (4)}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{\mu r^2 \sin^2\theta} \qquad \text{— (5)}$$

Combining (1) & (2), we have

$$\frac{1}{\mu} p_r \frac{dp_r}{dt} = \left(\frac{p_\phi^2}{\mu r^3} - \frac{dV}{dr}\right)\frac{dr}{dt}$$

only for 2 dimensions from (2)

$$\rightarrow \left(\frac{p_\phi^2}{\mu r^3} - \frac{dV}{dr}\right) dr - \frac{1}{\mu} p_r dp_r = 0$$

Integrate the above.

$$\frac{1}{2\mu}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + V(r) = E \text{ (constant)} \qquad \text{— (6)}$$

Combining (3) & (4), we have

$$p_\theta \dot{p}_\theta = \frac{p_\phi^2 \dot{\theta} \cos\theta}{\sin^3\theta}$$

Integrate.

for the sake of argument

$$p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} = l^2 \text{ (constant)} \qquad \text{— (7)}$$

From (6), solve for P_r.

$$P_r = \frac{1}{r}\sqrt{2\mu r^2(E-V) - P_\phi^2} \quad \text{or} \quad r^2 P_r = r\sqrt{2\mu r^2(E-V)-P_\phi^2}$$

From (7), solve for P_θ.

$$P_\theta = \sqrt{\ell^2 - \frac{P_\phi^2}{\sin^2\theta}}$$

Combining (1) & (3), we obtain

$$\frac{dr}{P_r r^2} = \frac{d\theta}{P_\theta}$$

Therefore,

$$\int \frac{dr}{r\sqrt{2\mu r^2(E-V)-P_\phi^2}} = \int \frac{d\theta}{\sqrt{\ell^2 - \frac{P_\phi^2}{\sin^2\theta}}} = \sin^{-1}\left(\frac{\ell\cos\theta}{\sqrt{\ell^2-P_\phi^2}}\right)$$

Combining (3) & (5), we obtain

$$\frac{d\phi}{P_\phi} = \frac{d\theta}{P_\theta \sin^2\theta}$$

From (7), solve for P_ϕ.

$$P_\phi = \sqrt{(\ell^2-P_\theta^2)\sin^2\theta}$$

Therefore,

$$\int \frac{d\phi}{\sqrt{(\ell^2-P_\theta^2)\sin^2\theta}} = \int \frac{d\theta}{\sin^2\theta\sqrt{\ell^2-\frac{P_\phi^2}{\sin^2\theta}}}$$

or $$\phi = \int \frac{P_\phi\, d\theta}{\sin^2\theta\sqrt{\ell^2 - P_\phi^2 \operatorname{cosec}^2\theta}}$$