Prob 8.4

A Lagrangian can be expressed as (2nd ed. Section 8.1)

$$L(q,\dot{q},t) = L_0(q,t) + \tilde{\dot{q}}\,\mathbb{a} + \frac{1}{2}\tilde{\dot{q}}\,\mathbb{T}\,\dot{q}$$

Thus, the Hamiltonian can be

$$H(q,p,t) = \frac{1}{2}(\tilde{\mathbb{p}} - \tilde{\mathbb{a}})\mathbb{T}^{-1}(\mathbb{p} - \mathbb{a}) - L_0(q,t)$$

The Lagrangian referred to in the problem can be written as

$$L = L_0 + \tilde{\dot{q}}\,\mathbb{a} + \frac{1}{2}\tilde{\dot{q}}\,\mathbb{T}\,\dot{q} \qquad L_0 = -k\sqrt{x^2+y^2}$$

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \quad \mathbb{a} = \begin{pmatrix} 0 \\ g \\ 0 \end{pmatrix}, \quad \mathbb{T} = \begin{pmatrix} 2a & c & fy^2 \\ c & \frac{2b}{x} & 0 \\ fy^2 & 0 & 0 \end{pmatrix}$$

Since $\mathbb{T}^{-1} = \dfrac{\tilde{\mathbb{T}}_c}{|\mathbb{T}|}$,

$$\mathbb{T}^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{fy^2} \\ 0 & \frac{x}{2b} & -\frac{x}{2bfy^2} \\ \frac{1}{fy^2} & -\frac{x}{2bfy^2} & \frac{cx^2 - 4ab}{2bf^2y^4} \end{pmatrix}$$

Thus, plug in the following formulation:

$$H = \frac{1}{2}(\tilde{\mathbb{p}} - \tilde{\mathbb{a}})\mathbb{T}^{-1}(\mathbb{p} - \mathbb{a}) - L_0 \qquad \mathbb{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$$= \frac{1}{2}\left[\frac{x(P_y - g)^2}{2b} + \frac{(c^2x - 4ab)P_z^2}{2bf^2y^4} + \frac{2P_x P_z}{fy^2} - \frac{x(P_y - g)P_z}{bfy^2} \right] + k\sqrt{x^2+y^2}$$

From the Lagrangian, z is obviously cyclic. Therefore, P_z is conserved, and so is the Hamiltonian.