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Derivation of the mean and variance with the posterior probability

$$\underset{\text{posterior}}{\pi(\mu|D)} \propto \underset{\text{likelihood}}{\exp\left[-\frac{(\bar{x}-\mu)^2}{2\frac{\sigma^2}{n}}\right]} \times \underset{\text{prior}}{\exp\left[-\frac{(\mu_0-\mu)^2}{2\sigma_0^2}\right]}$$

The exponetial part

$$= -\left[\frac{(\bar{x}-\mu)^2}{2\frac{\sigma^2}{n}} + \frac{(\mu_0-\mu)^2}{2\sigma_0^2}\right]$$

$$= -\frac{1}{2\sigma_0^2}\left[\frac{n\sigma_0^2}{\sigma^2}(\mu-\bar{x})^2 + (\mu-\mu_0)^2\right]$$

$$= -\frac{1}{2\sigma_0^2}\left[\frac{n\sigma_0^2}{\sigma^2}(\mu^2-2\mu\bar{x}+\bar{x}^2) + (\mu^2-2\mu_0\mu+\mu^2)\right]$$

$$= -\frac{1}{2\sigma_0^2}\left[\frac{n\sigma_0^2}{\sigma^2}\mu^2 - \frac{n\sigma_0^2}{\sigma^2}2\mu\bar{x} + \frac{n\sigma_0^2}{\sigma^2}\bar{x}^2 + \mu^2 - 2\mu_0\mu + \mu_0^2\right]$$

$$= -\frac{1}{2\sigma_0^2}\left[\left(\frac{n\sigma_0^2}{\sigma^2}+1\right)\mu^2 - \left(\frac{n\sigma_0^2}{\sigma^2}\bar{x}+\mu_0\right)2\mu + \left(\frac{n\sigma_0^2}{\sigma^2}+\mu_0^2\right)\right]$$

Express the inside bracket in a perfect square.

$$= -\left[\frac{1}{2\sigma_0^2}\left(\frac{n\sigma_0^2}{\sigma^2}+1\right)\left(\mu - \frac{\frac{n\sigma_0^2}{\sigma^2}\bar{x}+\mu_0}{\frac{n\sigma_0^2}{\sigma^2}+1}\right) + \{\text{constant}\}\right]$$

Therefore, the posterior μ_1 and σ_1 will be

$$\mu_1 = \frac{\frac{n\sigma_0^2}{\sigma^2}\bar{x}+\mu_0}{\frac{n\sigma_0^2}{\sigma^2}+1} = \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\sigma_0^2}}{\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}}$$

$$\sigma_1^2 = \frac{\sigma_0^2}{\frac{n\sigma_0^2}{\sigma^2}+1} = \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}}$$