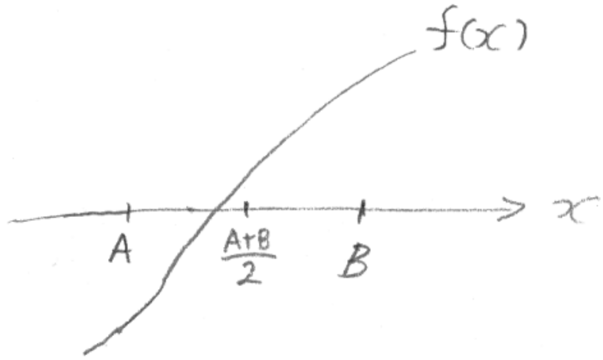
Solving non-linear/transcendental equations

- Bisection method



① Set A and B to the range.

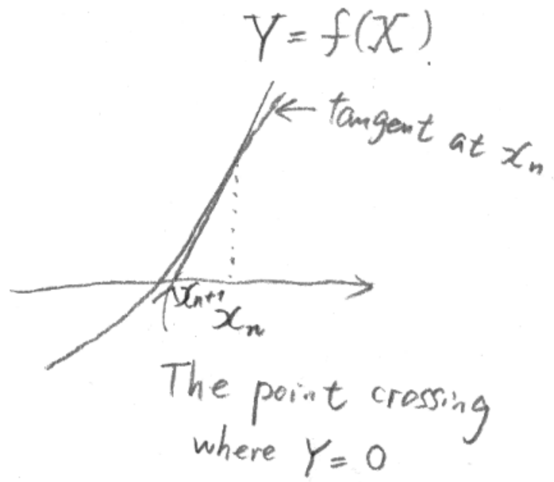
② Find $X = \frac{A+B}{2}$.

③ Calculate $f(X)$.

④ If $f(A)\cdot f(X)$ is negative, the value of X will become B (right).
If $f(A)\cdot f(X)$ is positive, the value of X will become A (left).

⑤ If the difference b/w A and B is very small, say, 10^{-10}, give the solution. Otherwise, go to the second step.

- Newton's method



The line equation of the tangent line is

$$Y - f(x_n) = f'(x_n)(X - x_n)$$

The point x_{n+1} will be obtained by letting $Y = 0$.

$$-f(x_n) = f'(x_n)(x_{n+1} - x_n)$$

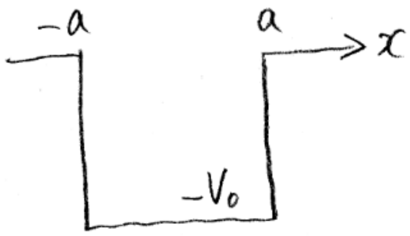
$$\Rightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This will be converged to the root.

If you substitute $f'(x_n)$ with $\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$, this is called secant method.

* Physics example of transcendental equation

⇨ Quantum mechanics with a square well potential



The bound-state solution:

$$\psi(x) = \begin{cases} C e^{\beta x} & \text{for } -\infty < x < -a \\ B \cos \alpha x & \text{for } -a < x < a \\ C e^{-\beta x} & \text{for } a < x < \infty \end{cases}$$

at the edge of the well, the wave function and its first derivative are equal (continuous).

$$B \cos \alpha a = C e^{-\beta a}$$

$$-\alpha B \sin \alpha a = -\beta C e^{-\beta a}$$

Thus,

$$\alpha a \tan \alpha a - \beta a = 0 \quad \text{——} \quad (\text{eq. } 1)$$

Recall $\alpha = \sqrt{\dfrac{2m(E+V_0)}{\hbar^2}}$; and $\beta = \sqrt{\dfrac{-2mE}{\hbar^2}}$.

Therefore eq. 1 is a transcendental equation of finding energy, E.