How to derive the Hamiltonian of a particle in the electromagnetic fields

The Lagrangian of Lorenz invariant electromagnetic fields becomes:

$$L(\vec{r},\vec{v},t) = -m_0c^2\sqrt{1-v^2/c^2} - e\phi(\vec{r},t) + e\vec{v}\vec{A}(\vec{r},t) \tag{1}$$

The first term in the Lagrangian, $-m_0c^2\sqrt{1-v^2/c^2}$, is consistent to derive the relativistically invariant Newton's equation.

Hamiltonian is described by the Legendre transformation of the Lagrangian:

$$H(\vec{r},\vec{p},t) = \sum_i p_i\dot{r}_i - L(\vec{r},\dot{\vec{r}},t) \tag{2}$$

The momentum is described with a Lagrangian:

$$p_i = \frac{\partial L}{\partial \dot{r}}$$

Using (1), we obtain

$$\vec{p} = \frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}} + e\vec{A} \tag{3}$$

The first term of right hand side of (3) is relativistic momentum:

$$\vec{p}_{\text{rel}} = \frac{m_0 v}{\sqrt{1-v^2/c^2}} \tag{3'}$$

Therefore, (3) is rewritten as

$$\vec{p}_{\text{rel}} = \vec{p} - e\vec{A} \tag{4}$$

The relativistic energy of a free particle is:

$$\varepsilon = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} \tag{5}$$

Eliminate v using (3)' and (5).

$$\varepsilon^2 = m_0^2c^4 + c^2p_{\text{rel}}^2 \tag{6}$$

Now, let us derive the Hamiltonian. Using (1), (2), and (3), we obtain

$$\begin{aligned} H &= \frac{m_0 v^2}{\sqrt{1-v^2/c^2}} + m_0 c^2 \sqrt{1-v^2/c^2} + e\phi \\ &= \frac{m_0 v^2 + m_0 c^2 \left(1-v^2/c^2\right)}{\sqrt{1-v^2/c^2}} + e\phi \\ &= \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} + e\phi \end{aligned} \tag{7}$$

Using (5) and (6), we have

$$H = \sqrt{m_0^2 c^4 + c^2 p_{\mathrm{rel}}^2} + e\phi \tag{8}$$

Substituting (4) into (8), we derive the final form of the Hamiltonian.

$$H = \sqrt{m_0^2 c^4 + c^2 \left(\vec{p} - e\vec{A}\right)^2} + e\phi$$