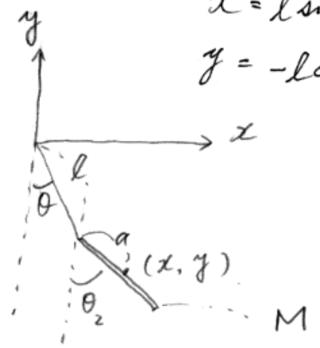

$x = l\sin\theta_1 + a\sin\theta_2$

$y = -l\cos\theta_1 - a\cos\theta_2$

example. Double pendulum

The second pendulum is a rigid body, so we have to consider the moment of inertia.

1. kinetic energy

$$T = \frac{1}{2}M(\dot{x}^2+\dot{y}^2) + \frac{1}{2}\cdot\left(\frac{Ma^2}{3}\right)\cdot\dot{\theta}_2^2$$

Make use of an approximation, $\sin\theta \sim \theta$, and $\cos\theta = 1-\frac{\theta^2}{2}$.

Take until the second order of θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$.

$$T \sim M\left(l^2\dot{\theta}_1^2/2 + al\dot{\theta}_1\dot{\theta}_2 + 2a^2\dot{\theta}_2^2/3\right)$$

2. potential energy

$$U = Mgy = (-l\cos\theta_1 - a\cos\theta_2)Mg \sim \frac{1}{2}(l\theta_1^2 + a\theta_2^2)Mg + \text{const.}$$

3. equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0 : \quad Ml^2\ddot{\theta}_1 + Mal\ddot{\theta}_2 + Mgl\theta_1 = 0$$

$$\rightarrow l\ddot{\theta}_1 + a\ddot{\theta}_2 + g\theta_1 = 0 \qquad \text{— (a1)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = 0 : \quad Mal\ddot{\theta}_1 + \frac{4}{3}Ma^2\ddot{\theta}_2 + Mga\theta_2 = 0$$

$$\rightarrow l\ddot{\theta}_1 + \frac{4}{3}a\ddot{\theta}_2 + g\theta_2 = 0 \qquad \text{— (a2)}$$

4. To solve (a1) and (a2), we should put

$\theta_1 = Ae^{i\omega t}$; $\theta_2 = Be^{i\omega t}$.

Then, we obtain

$$\left.\begin{aligned}(g-l\omega^2)A - a\omega^2 B &= 0\\ -l\omega^2 A + \left(g-\frac{4}{3}a\omega^2\right)B &= 0\end{aligned}\right\} \qquad \text{— (b)}$$

The solutions should not be $A=B=0$. So

$$\begin{vmatrix} g-l\omega^2 & -a\omega^2 \\ -l\omega^2 & g-\frac{4}{3}a\omega^2 \end{vmatrix} = 0.$$

Thus,

$$al\omega^4 - g(3l+4a)\omega^2 + 3g^2 = 0$$

The solutions for ω^2 are:

$$\omega^2 = \frac{3l+4a \pm \sqrt{(3l+4a)^2-12al}}{2al}\, g$$

$$ax^2+bx+c=0 \;\rightarrow\; x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

The inside of the root is a positive quantity. So the two solutions of ω^2 should be positively real. The two solutions are put as ω_1^2 and ω_2^2. According to (b), we can determine the ratio.

$$\frac{A_1}{B_1} = \frac{a\omega_1^2}{g-l\omega_1^2} \;;\quad \frac{A_2}{B_2} = \frac{a\omega_2^2}{g-l\omega_2^2}$$

Assuming that C_1 and C_2 are arbitrary constants, we have

$$A_1 = a\omega_1^2 C_1$$
$$B_1 = (g-l\omega_1^2)C_1$$
$$A_2 = a\omega_2^2 C_2$$
$$B_2 = (g-l\omega_2^2)C_2$$

Therefore we obtain

$$\theta_1 = A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t}$$
$$= a\omega_1^2 C_1 e^{i\omega_1 t} + a\omega_2^2 C_2 e^{i\omega_2 t}$$
$$\theta_2 = B_1 e^{i\omega_1 t} + B_2 e^{i\omega_2 t}$$
$$= (g-l\omega_1^2)C_1 e^{i\omega_1 t} + (g-l\omega_2^2)C_2 e^{i\omega_2 t}$$