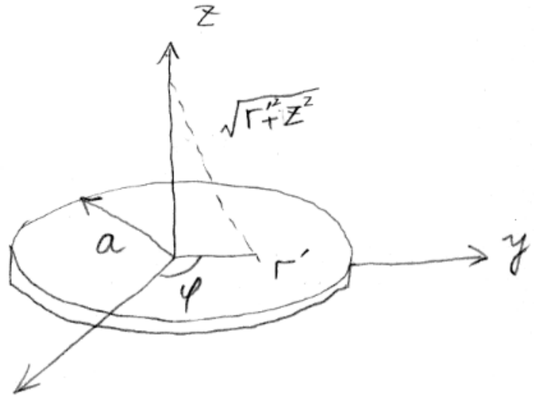
The electric field and potential for a charged disk

Find the potential V at height z above the center of a uniformly charged disk of radius 'a'.



Solution : From Gauss's law,

The charge density is $\rho = \begin{cases} \frac{Q}{\pi a^2}\delta(z') & \text{for } x'^2 + y'^2 \leq a^2 \\ 0 & \text{elsewhere} \end{cases}$

The potential above the center of the plate is then

$$V(0,0,z) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r})\, d^3r'}{|\vec{r}-\vec{r}'|}$$

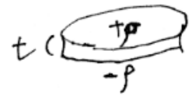
distance between a point on z-axis and all surface of the plate.

$$= \frac{1}{4\pi\epsilon_0}\int_0^a\int_0^{2\pi} \frac{Q}{\pi a^2}\frac{r'dr'd\phi'}{\sqrt{r'^2+z^2}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{\pi a^2}\int_0^{2\pi} d\phi \int_0^a \frac{r'dr'}{\sqrt{r'^2+z^2}}$$

$$= \frac{Q}{2\pi a^2\epsilon_0}\int_0^a r'(r'^2+z^2)^{-\frac{1}{2}}dr'$$

* If the plate has a finite width, like 't', the charge density will be $\rho = \frac{Q}{\pi a^2}t$.



Put $r'^2 = x$. $\rightarrow$ $dr' = \frac{1}{2r'}dx$ $[0 \sim a^2]$

$$= \frac{Q}{2\pi a^2\epsilon_0}\int_0^{a^2}\frac{1}{2}(x+z^2)^{-\frac{1}{2}}dx$$

$$= \frac{Q}{2\pi a^2\epsilon_0}\Big[(x+z^2)^{\frac{1}{2}}\Big]_0^{a^2}$$

$$= \frac{Q}{2\pi a^2\epsilon_0}\left(\sqrt{a^2+z^2} - z\right)$$

Therefore, the electric field is

$$E_z(0,0,z) = -\frac{\partial V}{\partial z} = \frac{Q}{2\pi a^2\epsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$