EM potential

- Pick out "monopole" equation in Maxwell's eqs.

$$\vec{\nabla}\cdot\vec{B} = 0.$$

Form that, we find

$$\vec{B} = \vec{\nabla}\times\vec{A}. \qquad \text{—— ①}$$

- Pick out Faraday's Law,

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}.$$

Plug ① into above.

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}).$$

Therefore $\vec{E} = -\frac{\partial\vec{A}}{\partial t}$, but $\vec{\nabla}\times\nabla\phi = \vec{0}$.

As a result,

$$\vec{E} = -\frac{\partial A}{\partial t} - \mathrm{grad}\,\phi. \qquad \text{—— ②}$$

① and ② are called EM potential.

$$\begin{cases} \vec{B} = \vec{\nabla}\times\vec{A} \\ \vec{E} = -\frac{\partial\vec{A}}{\partial t} - \mathrm{grad}\,\phi \end{cases}$$

However, $\vec{A}$ and ϕ have arbitrariness to obtain the same $\vec{B}$ and $\vec{E}$.

From ①,

$$\underline{A' = A + \mathrm{grad}\,\chi} \quad \text{— ③}$$ (χ is an arbitrary scalar potential.)

Plug A' into ②.

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \frac{\partial}{\partial t}\mathrm{grad}\,\chi - \mathrm{grad}\,\phi$$

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To get the same $\vec{E}$, the RHS should be

$$-\frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t}\,\mathrm{grad}\,\chi - \mathrm{grad}\,\phi = -\frac{\partial \vec{A}'}{\partial t} - \mathrm{grad}\,\phi'.$$

We can determine $\phi' = \phi - \frac{\partial \chi}{\partial t}$. ——— ④

③ and ④ are called Lorentz gauge.

* Sometimes ϕ can be expressed as V.

In static current, the Ampére's law is

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}. \quad \text{———} ⑤$$

Recall the magnetic field can be expressed as

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

Take rotation for the both sides of above.

$$\nabla \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A}$$
$$= \nabla(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}.$$

Here we employ Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, and ⑤.

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \quad \text{———} ⑥$$

It looks like Poisson's eq.,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

So we can express ⑥ as

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{r}\, dv.$$