Fibonacci Sequence / Fibonacci Numbers

■ The basics

The Fibonacci numbers are expressed by the following recursion equation:

$$\begin{cases} F_1 = F_2 = 1 \\ F_{n+2} = F_{n+1} + F_n \end{cases} \tag{1}$$

The series of numbers are 1, 1, 2, 3, 5, 8, 13, 21, … You can derive a rule such that a Fibonacci number is the sum of the previous and before previous numbers.:

$$F_n = F_{n-2} + F_{n-1} \tag{1'}$$

■ The period of Fibonacci sequence

One can generate specific periods divided the sequence by prime numbers except 5. For example, when the sequence divided by 2, we have the period of three:

1, 1, 0, 1, 1, 0, 1, 1, 0…

When divided by 3 and 11, we have the periods 8 and 10, respectively.

■ Cassini's identity

Fibonacci sequence also has the following property:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^{n+1}$$

■ Other properties

$$\sum_{n=1} F_n^2 = F_n F_{n+1}$$

$$\sum_{n=1} (F_n F_{n+1})^3 = (F_n F_{n+1} F_{n+2})^2 / 4$$

$$F_{2n+1} = F_{n-1}F_m + F_n F_{m+1}$$

$$F_{2n+1} = F_{n+1}^2 + F_n^2$$

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2$$

$$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$$

■ The solution

Let us express the sequence as

$$F_{n+2} = (p+q)F_{n+1} - pqF_n \tag{2}$$

Thus, $p+q=1$ and $pq=-1$ compared with the Fibonacci sequence, (1). From the coefficients relationship for a quadratic equation, one can set up the equation,

$x^2 - x - 1 = 0$, and the solutions are $p, q = \left(1 \pm \sqrt{5}\right)/2$.

Equation (2) can be modified as

$$F_{n+2} - pF_{n+1} = q\left(F_{n+1} - pF_n\right) \quad (3)$$

This is a geometric sequence of $F_{n+1} - pF_n$ with the initial term, $F_2 - pF_1$, and the common ratio, q. Therefore, the general terms are

$$F_{n+1} - pF_n = \left(F_2 - pF_1\right)q^{n-1} = \left(1 - p\right)q^{n-1} \quad (4)$$

Substituting the values of p and q, we have

$$F_{n+1} - \frac{1+\sqrt{5}}{2}F_n = \left[1 - \frac{1+\sqrt{5}}{2}\right]\left[\frac{1-\sqrt{5}}{2}\right]^{n-1} = \left[\frac{1-\sqrt{5}}{2}\right]^n \quad (5)$$

or

$$F_{n+1} - \frac{1-\sqrt{5}}{2}F_n = \left[1 - \frac{1-\sqrt{5}}{2}\right]\left[\frac{1+\sqrt{5}}{2}\right]^{n-1} = \left[\frac{1+\sqrt{5}}{2}\right]^n \quad (6)$$

Calculate (6) – (5). Then we have the formula for Fibonacci numbers.

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \quad (7)$$

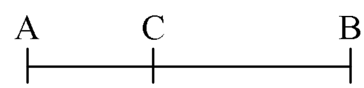
■ The ratio between terms

The limit of ratio of Fibonacci terms are given as follows:

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} = 1.6176\ldots \quad (8)$$

■ Golden ratio 1

From the following segment, the golden ratio defines: $AC : CB = CB : AB$. When $AC = 1$ and $CB = x$, we have



$$1 : x = x : \left(1 + x\right) \quad (9)$$

Namely,

$$x^2 - x - 1 = 0 \quad (10)$$

The positive solution is $x = \left(1 + \sqrt{5}\right)/2$.

■ Golden ratio 2

From (9), we can also have the following relationship:

$$x = 1 + \frac{1}{x} \tag{11}$$

Substitute *x* into (11) to have the following continued fraction:

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}} = \frac{1+\sqrt{5}}{2} \tag{12}$$

From (10), we can derive

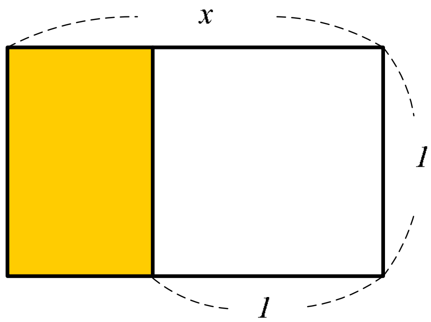
$$x = \sqrt{1+x} \tag{13}$$

Likewise, we can have a nested radical:

$$x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}} = \frac{1+\sqrt{5}}{2} \tag{14}$$

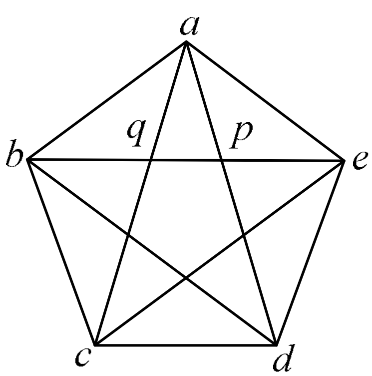
■ Golden ratio 3

When the original rectangle and the other one divided by the square are similar as shown, the ratio, 1 : *x*, is equal to $(1 + \sqrt{5})/2$.



For an equilateral pentagon shown, we have golden ratios:

$$\frac{bp}{pe} = \frac{ep}{pq} = \frac{ae}{be} = \frac{ec}{ab} = \frac{1+\sqrt{5}}{2}$$



■ Extra knowledge 1

The following is called Lucas sequence:

$$\begin{cases} L_1 = 2, \ \ L_2 = 1 \\ L_{n+2} = L_{n+1} + L_n \end{cases}$$

Some of the properties are

$$L_n = F_{n+1} + F_{n-1}$$

$$F_{2n} = F_n L_n$$

■ Extra knowledge 2

When the original rectangle and the other one divided into half as shown, we have $x = \sqrt{2}$. This is called the silver ratio.