General Angular Momentum

Definition : $J_\pm = J_x \pm i J_y$

We have to know the commutation relations.

$$[J_z, J_\pm] = [J_z, J_x \pm i J_y]$$
$$= [J_z, J_x] \pm [J_z, iJ_y]$$
$$= i\hbar J_y \pm \hbar J_x$$
$$= \hbar(iJ_y \pm J_x)$$
$$= \pm\hbar J_\pm \qquad \text{— (A)}$$

$$[J^2, J_\pm] = 0 \qquad \text{— (B)}$$

Operate (A) and (B) to simultaneous eigen functions ψ_{JM}.

$$[J_z, J_\pm]\psi_{JM} = (J_z J_\pm - J_\pm J_z)\psi_{JM} = \pm\hbar J_\pm \psi_{JM}$$

$$J_z J_\pm \psi_{JM} - m\hbar J_\pm \psi_{JM} = \pm\hbar J_\pm \psi_{JM}$$

$$\Rightarrow J_z J_\pm \psi_{JM} = (m \pm 1)\hbar J_\pm \psi_{JM} \qquad \text{— (C)}$$

$$[J^2, J_\pm]\psi_{JM} = (J^2 J_\pm - J_\pm J^2)\psi_{JM} = 0$$

$$\Rightarrow J^2 J_\pm \psi_{JM} = j(j+1)\hbar^2 J_\pm \psi_{JM} \qquad \text{— (D)}$$

(C) and (D) indicate that $J_\pm \psi_{JM}$ is only constant multiplication of $\psi_{J\,M\pm1}$.

To get the const.,

$$\langle (J_\pm \psi_{JM})^* | J_\pm \psi_{JM} \rangle = \langle \psi^*_{JM} | J_\mp J_\pm | \psi_{JM} \rangle = \langle \psi^*_{JM} | (J^2 - J_z^2 \mp \hbar J_z) | \psi_{JM} \rangle$$
$$= [J(J+1) - M(M\pm1)]\hbar^2$$

Therefore, $J_\pm \psi_{JM} = \hbar\sqrt{J(j+1) - m(M\pm1)}\, \psi_{J\cdot M\pm1}$.