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Mathematical Physics

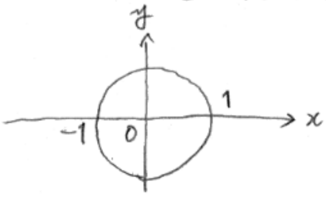
The (infinite) integral of complex variables

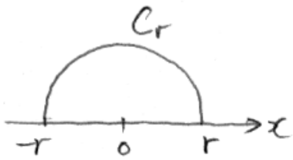
$$\int_0^{2\pi} \frac{d\theta}{1-2c\cos\theta+c^2} \quad |c|<1$$

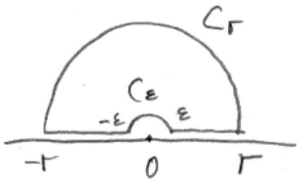
$$\int_0^{\infty} \frac{\cos mx}{x^2+a^2}\,dx$$

$$\int_0^{\infty} \frac{\sin x}{x}\,dx$$

① Find the contours :







② Rewrite the integrant in terms of complex variables :

$$\left\{ \begin{array}{ll} z = e^{i\theta} & d\theta = \dfrac{dz}{iz} \\ \cos\theta = \dfrac{(z+z^{-1})}{2} & \end{array} \right\}$$

$$I = \int_{|z|=1} \frac{dz/iz}{1-c(z+z^{-1})+c^2}$$

$$= \int_{|z|=1} \frac{i\,dz}{c(z-c)(z-c^{-1})}$$

$x \to z$

$\cos mx \to e^{imx}$

$$I = \int_C \frac{e^{imz}}{z^2+a^2}\,dz$$

$x \to z$

$\sin x \to e^{iz}$

$$I = \int_C \frac{e^{iz}}{z}\,dz$$

③ Divide the integral into each contour.

$$I = \int_{-r}^{r} \frac{e^{imx}}{x^2+a^2}\,dx + \int_{C_r} \frac{e^{imz}}{z^2+a^2}\,dz$$

$$I = \int_{-r}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz + \int_{\varepsilon}^{r} \frac{e^{ix}}{x}\,dx + \int_{C_r} \frac{e^{iz}}{z}\,dz$$

④ Scrutinize the each term of the integrals :

$$I = \int_{C_r} \frac{e^{imz}}{z^2+a^2}\,dz$$

When $|z| \to \infty$, it becomes zero.

$$I = \int_{C_r} \frac{e^{iz}}{z}\,dz$$

When $|z| \to \infty$, it becomes zero.

⑤ Find the singular points in the contours :

$z = c$

$z = ia$

no pole

when $\varepsilon \to 0$, z is the half pole.

⑥ Calculate the residues :

$$\mathrm{Res}[f, c] = \lim_{z\to c}(z-c)f = \lim_{z\to c}\frac{i}{c(z-c^{-1})} = \frac{i}{c^2-1}$$

$$\mathrm{Res}[f, ia] = \lim_{z\to ia}(z-ia)f = \lim_{z\to ia}\frac{e^{imz}}{z+ia} = \frac{e^{-ma}}{2ia}$$

$$I = \int_{C_r} = 0$$

⑦ Obtain the final results by multiplying them by $2\pi i$.

$$I = 2\pi i \frac{i}{c^2-1} = \frac{2\pi}{1-c^2}$$

$$I = 2\pi i \frac{e^{-ma}}{2ia} = \frac{\pi e^{-ma}}{a}$$

But

$$\int_0^\infty \frac{\cos mx}{x^2+a^2}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos mx}{x^2+a^2}dx = \frac{1}{2}\mathrm{Re}\left(\int_{-\infty}^{\infty}\frac{e^{imx}}{x^2+a^2}dx\right)$$

$$\therefore \int_0^\infty \frac{\cos mx}{x^2+a^2}dx = \frac{\pi e^{-ma}}{2a}$$

$$I = \int_{-r}^{-\varepsilon}\frac{e^{ix}}{x}dx + \int_{k_\varepsilon}\frac{e^{iz}}{z}dz + \int_\varepsilon^r \frac{e^{ix}}{x}dx = 0$$

$$I = \int_\varepsilon^r \frac{e^{ix}-e^{-ix}}{x} + \underline{\int_{C_\varepsilon}\frac{e^{iz}}{z}dz}$$

half circle, negative direction around pole $z = 0$. So, the integral is $\frac{1}{2}\cdot\{-2\pi i\,\mathrm{Res}[f, 0]\}$

$= -\pi i$

And $\frac{e^{ix}-e^{-ix}}{x} = 2i\frac{\sin x}{x}$,

therefore, when $r\to\infty$, $\varepsilon\to 0$,

$$I = 2i\int_0^\infty \frac{\sin x}{x}dx - \pi i = 0$$

$$\therefore \int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$$