Laplace's eq. for Plane Polar Coordinates

In plane polar coordinates, $\nabla^2 V = 0$ will be

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

But $V = R(r)\Phi(\varphi)$, Divide the both sides by V.

$$\frac{r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)}{R} = -\frac{\frac{\partial^2 \Phi}{\partial \varphi^2}}{\Phi} = \text{Constant} = m^2$$

Therefore

$$\frac{\partial^2 \Phi}{\partial \varphi^2} = -m^2 \Phi .$$

That one is solved as

$$\Phi = \alpha \cos m\varphi + \beta \sin m\varphi \qquad (m \text{ is an integer.})$$

The remaining equation

$$r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = m^2 R$$

We can try this function $R = Cr^l$.

$$LHS = l^2 C r^l \quad ; \quad RHS = m^2 R$$
$$= l^2 R$$

$$\Rightarrow \quad l^2 R = m^2 R$$

$$\Rightarrow \quad l = \pm m$$

Therefore, $R = \gamma r^m + \delta r^{-m}$.

That solution is for $m \neq 0$.

For $m=0$,

$$\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)=0 \qquad (r\neq 0)$$

$$\Rightarrow \quad r\frac{\partial R}{\partial r}=\alpha$$

$$\Rightarrow \quad \frac{\partial R}{\partial r}=\frac{\alpha}{r}$$

$$\therefore \quad R=\alpha \ln r+\beta .$$

We have obtained

when $m\neq 0$:

$$\Phi=\sum_{m=1}^{\infty}\left(\alpha_m\cos m\varphi+\beta_m\sin m\varphi\right)$$

$$R=\sum_{m=1}^{\infty}\left(\gamma_m r^m+\frac{\delta_m}{r^m}\right)$$

when $m=0$:

$$R=\alpha_0+\beta_0\ln r .$$

Combining all, we get

$$V(r,\varphi)=\alpha_0+\beta_0\ln r+\sum_{m=1}^{\infty}\left(\gamma_m r^m+\frac{\delta_m}{r^m}\right)\left(\alpha_m\cos m\varphi+\beta_m\sin m\varphi\right).$$

It is the general solution.