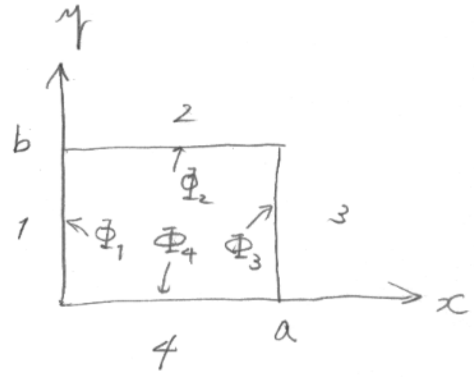
Static Potentials in Vacuum [Laplace's equation]

① Cartesian Coordinates in 2-D.



(a) Follow the figure.

(b) Solve $\nabla^2 V(x, y) = 0$

(But in this case, $\Phi_1 = \Phi_2 = \Phi_3 = 0$,
and $\Phi_4 = \Phi_4(x)$)

(c) The solution will be superposed.

The potential is expressed as

$$V_4(x, y) = X(x)\, Y(y).$$

We solve

$$\nabla^2 V_4 = 0.$$

Dividing the both sides by $V_4(x, y)$, we get

$$\frac{\nabla^2 V_4}{V_4} = \frac{\frac{\partial^2 V_4}{\partial x^2} + \frac{\partial^2 V_4}{\partial y^2}}{V_4} = \frac{Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2}}{XY}$$

$$\therefore \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

On the left of this last eq., we have a function depending on x only, which, according to the right side, is independent of x; therefore it must be constant.

$$\left.\begin{aligned} \frac{1}{X}\frac{\partial^2 X}{\partial x^2} &= -\lambda^2 \quad \Rightarrow \quad \frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0 \\ -\lambda^2 &= -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} \quad \Rightarrow \quad \frac{\partial^2 Y}{\partial y^2} - \lambda^2 Y = 0 \end{aligned}\right\}$$

The general solution is

$$\begin{cases} X = \alpha \cos \lambda x + \beta \sin \lambda x \\ Y = \gamma \cosh \lambda y + \delta \sinh \lambda y \end{cases}$$

Apply the boundary conditions.

i) for $x = 0$, $\rightarrow \Phi_1$

$$V_4(0, y) = \alpha(\gamma \cosh \lambda y + \delta \sinh \lambda y) = \Phi_1(y) = 0$$

Here we can choose either

$$\alpha = 0, \quad \gamma = \delta = 0$$

for the solution.

However, if we take $\gamma = \delta = 0$, it will be trivial.

So take $\alpha = 0$.

ii) for $x = a$, $\rightarrow \Phi_3$

$$V_4(a, y) = \beta \sin \lambda a (\gamma \cosh \lambda y + \delta \sinh \lambda y) = \Phi_3(y) = 0$$

If we take $\beta = 0$ or $\gamma = \delta = 0$, it will be trivial.

So we take $\sin \lambda a = 0$, which means

$$\lambda a = n\pi \rightarrow \lambda = \frac{n\pi}{a} \quad (n = \text{an integer})$$

iii) for $y = b$ $\rightarrow \Phi_2$

$$V_4(x, b) = \beta \sin \frac{n\pi x}{a}\left(\gamma \cosh \frac{n\pi b}{a} + \delta \sinh \frac{n\pi b}{a}\right) = \Phi_2(x) = 0$$

In this time, we can determine $\delta = -\gamma \coth \frac{n\pi b}{a}$ so that $V_4 = 0$.

iv) for $y = 0$ $\rightarrow \Phi_4$

$$V_4(x, 0) = \beta\gamma \sin \frac{n\pi x}{a} \quad \left(\because \begin{array}{l} \sinh 0 = 0 \\ \cosh 0 = 1 \end{array}\right)$$

$$= \Phi_4$$

We just put $\beta\gamma \rightarrow \gamma$ to avoid unnecessary constants.

Express it in a linear combination as

$$V_4(x, 0) = \Phi_4(x) = \sum_{n=1}^{\infty} \gamma_n \sin \frac{n\pi x}{a}.$$

We immediately recognize the equation above as the Fourier expansion of $\Phi_4(x)$, allowing us to write

$$\gamma_n = \frac{2}{a}\int_0^a \Phi_4(x) \sin\frac{n\pi x}{a}\, dx$$

Gathering terms, we obtain

$$V_4(x, y) = \sum_{n=1}^{\infty} \gamma_n \sin\frac{n\pi x}{a}\left(\cosh\frac{n\pi y}{a} - \coth\frac{n\pi b}{a} \sinh\frac{n\pi y}{a}\right).$$