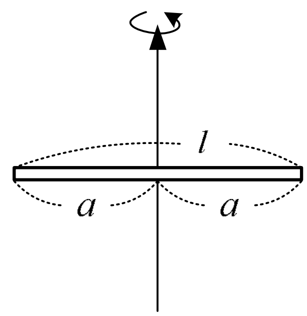
How to derive moments of inertia

Case 1: Rod



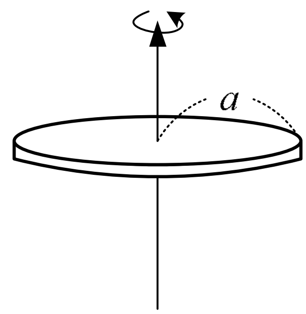
$$I = \int_{-a}^{a} \rho x^2$$

$$= \left[\frac{2}{3}\rho x^3\right]_{-a}^{a}$$

$$= \frac{a^2}{3}M \quad (\because M = \rho 2a)$$

$$= \frac{\ell^2}{12}M \quad (\because \ell = 2a)$$

Case 2: Disk



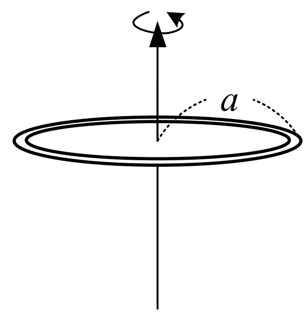
$$I = \int_0^{2\pi} x d\theta \int_0^a \rho x^2 dx$$

$$= \rho\left[\frac{x^4}{4}\right]_0^a \int_0^{2\pi} d\theta$$

$$= 2\pi\rho\frac{a^4}{4}$$

$$= \frac{a^2}{2}M \quad (\because M = \pi a^2 \rho)$$

Case 2: Ring (Rotating about *z* axis.)



$$I_z = \int_0^{2\pi} \rho a^2 \; a\, d\theta \qquad \text{(Because } a \text{ is fixed.)}$$

$$= \rho a^3 \int_0^{2\pi} d\theta$$

$$= 2\pi\rho a^3$$

$$= a^2 M \quad (\because M = 2\pi a \rho)$$

Since $I_z = I_x + I_y$ and assuming that the rotation axes are centered, $I_x = I_y = \frac{1}{2}a^2 M$.

Case 3: Solid Sphere

$I = \int \rho (x^2 + y^2) dxdydz$

For a polar coordinate,

$x = r \sin\theta \cos\phi$

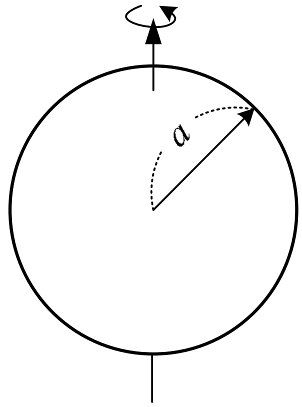
$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

Therefore,

$dxdydz = r^2 \sin\theta \, dr d\theta d\phi$ and

$x^2 + y^2 = r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi = r^2 \sin^2\theta$



$$I = \int_0^a \int_0^\pi \int_0^{2\pi} \rho r^4 \sin^3\theta \, dr d\theta d\phi$$

$$= \rho \left[\frac{r^5}{5} \right]_0^a \cdot \frac{4}{3} \cdot 2\pi$$

Because

$$\int_0^\pi \sin^3\theta \, d\theta = \int_0^\pi \sin^2\theta \sin\theta \, d\theta = \int_0^\pi (1 - \cos^2\theta) \sin\theta \, d\theta$$

Let $x = \cos\theta$ and $dx = -\sin\theta \, d\theta$.

$$= \int_1^{-1} (x^2 - 1) dx = \frac{4}{3}$$

The moment of inertia is

$$I = \rho \frac{8a^5}{15} \pi$$

$$= \frac{2}{5} a^2 M \qquad \because \left(M = \frac{4}{3} \pi a^3 \rho \right)$$

Case 3: Hollow Sphere

From the above case, consider the integral from radius *b* to *a*:

$$I = \int_b^a \int_0^\pi \int_0^{2\pi} \rho r^4 \sin^3\theta \, dr d\theta d\phi$$

This gives

$$I = \frac{2(a^5 - b^5)}{5(a^3 - b^3)} M$$

Take the limit, $b \to a$.

$$I = \lim_{b \to a} \frac{2(a^5 - b^5)}{5(a^3 - b^3)} M = \lim_{b \to a} \frac{2 \cdot 5b^4}{5 \cdot 3b^2} M = \frac{2}{3} a^2 M$$