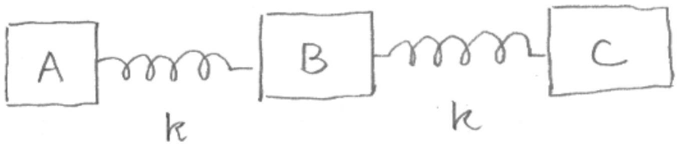
Normal Modes



Mass: A = B = C

Find the nomal modes of this motion.

① Potential energy :

$$V = \frac{1}{2}k(\eta_1-\eta_2)^2+\frac{1}{2}k(\eta_2-\eta_3)^2$$

② Kinetic energy :

$$T = \frac{1}{2}m(\dot\eta_1^2+\dot\eta_2^2+\dot\eta_3^2)$$

③ Lagrangian :

$$T-V = \frac{1}{2}\left[m(\dot\eta_1^2+\dot\eta_2^2+\dot\eta_3^2)-k\left\{(\eta_1-\eta_2)^2+(\eta_2-\eta_3)^2\right\}\right]$$

$$= \frac{1}{2}\left[m(\dot\eta_1^2+\dot\eta_2^2+\dot\eta_3^2)-k(\eta_1^2+2\eta_2^2+\eta_3^2-\eta_1\eta_2-\eta_2\eta_1-\eta_2\eta_3-\eta_3\eta_2)\right]$$

Therefore, we can write down

$$L = \frac{1}{2}(T_{ij}\dot\eta_i\dot\eta_j - V_{ij}\eta_i\eta_j) \qquad \text{———} \qquad (\alpha)$$

where

$$T_{ij} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}, \qquad V_{ij} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

④ Lagrange equation :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\eta} - \frac{\partial L}{\partial \eta} = 0$$

$$\therefore \quad T\ddot\eta + V\eta = 0 \qquad (\text{used } (\alpha))$$

⑤ Rewrite the vector $\vec{\eta}$ in terms of a wave formura :

- Actually the normal modes are collective motions where all three blocks move with the same frequency.

$$\vec{\eta}(t) = \vec{a}\, e^{i\omega_j t}$$

⑥ Plug above into the equation of motion :

$$(V - \omega_j^2 T)a_j = 0$$

But $\vec{a}$ is time-independent.

⑦ Find the nontrivial slution to exist :

$$\det[V - \omega_j^2 T] = 0$$

$$\begin{vmatrix} k-\omega_j^2 m & -k & 0 \\ -k & 2k-\omega_j^2 m & -k \\ 0 & -k & k-\omega_j^2 m \end{vmatrix} = 0$$

$$(k-\omega_j^2 m)\left\{(2k-\omega_j^2 m)(k-\omega_j^2 m) - k^2\right\} + k\left\{(-k)(k-\omega_j^2 m)\right\} = 0$$

$$(2k-\omega^2 m)(k-\omega^2 m)^2 - k^2(k-\omega^2 m) - k^2(k-\omega^2 m) = 0$$

$$(2k-\omega^2 m)(k-\omega^2 m)^2 - 2k^2(k-\omega^2 m) = 0$$

$$(k-\omega^2 m)\left\{\cancel{2k^2} - 2k\omega^2 m - k\omega^2 m + \omega^4 m^2 - \cancel{2k^2}\right\} = 0$$

$$\omega^2(k-\omega^2 m)(\omega^2 m^2 - 3km) = 0$$

Therefore,

$$\omega_1^2 = 0, \quad \omega_2^2 = \frac{k}{m}, \quad \omega_3^2 = \frac{3k}{m}$$

These are the frequency.

⑧ In order to find the normal modes, calculate $\vec{a}$:

Use the equation $(V - \omega_j^2 T)a_j = 0$. Substituting the three frequency into there, we get

$$\left\{ \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} - \begin{bmatrix} \omega_j^2 m & 0 & 0 \\ 0 & \omega_j^2 m & 0 \\ 0 & 0 & \omega_j^2 m \end{bmatrix} \right\} \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{bmatrix} = 0 \quad \text{——— ①}$$

Note:

You got $\overleftrightarrow{V}\vec{a}_k = \lambda_k \overleftrightarrow{T}\vec{a}_k$ (α). Especially, $\lambda = \omega_j^2$ in this case. The transversed equation will be

$$a_l^\dagger V = \lambda_l^* a_l^\dagger T \quad (\beta)$$

Multiply (α) by $a^\dagger$, and (β) by a. Then subtract each other.

$$0 = (\lambda_k - \lambda_l^*) a_l^\dagger T a_k$$

If λ_k and λ_l are real, the eq. will be

$$(\lambda_k - \lambda_l)\, a_l^\dagger T a_k = 0.$$

The solution, when $l \neq k$, is $a_l^\dagger T a_k = 0$.
When $l = k$, the term $a_l^\dagger T a_k$ is arbitrary, but to avoid the arbitrariness, we choose

$$a_k^T T a_k = 1. \quad (\gamma)$$

This is actually called normarization.

To get the normalization, use (γ); namely,

$$m(a_{i1}^2 + a_{i2}^2 + a_{i3}^2) = 1.$$

∗ note the suffixes!

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From ①,

$$\begin{cases} (k - \omega_j^2 m)\, a_{1j} - k\, a_{2j} = 0 \\ -k\, a_{1j} + (2k - \omega_j^2 m)\, a_{2j} - k\, a_{3j} = 0 \\ -k\, a_{2j} + (k - \omega_j^2 m)\, a_{3j} = 0 \end{cases}$$

when $\omega^2 = 0$,

$$a_{1j} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

when $\omega^2 = \frac{k}{m}$,

$$\begin{cases} -k a_{2j} = 0 \\ -k a_{1j} + k a_{2j} - k a_{3j} = 0 \\ -k a_{2j} = 0 \end{cases}$$

$$a_{2j} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

when $\omega^2 = \frac{3k}{m}$,

$$\begin{cases} -2k a_{1j} - k a_{2j} = 0 \\ -k a_{1j} - k a_{2j} - k a_{3j} = 0 \\ -k a_{2j} - 2k a_{3j} = 0 \end{cases}$$

$$a_{3j} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{matrix} 1 \\ -2 \\ 1 \end{matrix}$$

According to normalization, we obtain

$$a_{1j} = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad a_{2j} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad a_{3j} = \frac{1}{\sqrt{6m}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$