1- dimensional perturbation theory

System : S. H. O. $H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$

perturbation term : $\lambda H' = bx$

a. Calculate the energy shifts from the ground state.

Using

$$\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\delta_{n',n+1} + \sqrt{n}\,\delta_{n',n-1}\right),$$

we calculate the matrix elements of bx.

$$\langle n'|bx|n\rangle = b\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\delta_{n',n+1} + \sqrt{n}\,\delta_{n',n-1}\right)$$

For the first order perturbation,

$$\langle 0|bx|0\rangle = 0.$$

For the second order perturbation,

$$\langle 0|bx|1\rangle = b\sqrt{\frac{\hbar}{2m\omega}},$$

$$\langle 1|bx|0\rangle = b\sqrt{\frac{\hbar}{2m\omega}}.$$

This system is 1- particle and 1-dimensional, so non-degenerate. Therefore, the energy shift is

$$\Delta_0^{(2)} = \frac{|\langle 0|bx|1\rangle^* \langle 1|bx|0\rangle|}{E_0^{(0)} - E_1^{(0)}}$$

$$= \frac{\frac{b^2\hbar}{2m\omega}}{\frac{1}{2}\hbar\omega - \frac{3}{2}\hbar\omega} = \frac{\frac{b^2\hbar}{2m\omega}}{-\hbar\omega} = -\frac{b^2}{2m\omega^2}$$

b. Obtain the exact solution.

The total hamiltonian is :

$$H = \frac{p^2}{2m} + \frac{1}{2}mx^2\omega^2 + bx$$

Put $\frac{1}{2}m\omega^2$ as a.

$$H = \frac{p^2}{2m} + ax^2 + bx$$

$$= \frac{p^2}{2m} + a\left(x^2 + \frac{b}{a}x\right)$$

$$= \frac{p^2}{2m} + a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right\}$$

$$= \frac{p^2}{2m} + a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}$$

Therefore, we have

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x + \frac{b}{m\omega^2}\right)^2 - \frac{b^2}{2m\omega^2}.$$

According to the result, the x-coördinate is shifted by $-\frac{b}{m\omega^2}$, and the enegy is lowered by $\frac{b^2}{2m\omega^2}$.