Perturbation Theory (time independent)

2-particle, 1-dimensional, Simple Harmonic Oscillator.

$$H(x_1, x_2) = H(x_1) + H(x_2) + J x_1 x_2$$

where $H(x) = \frac{p^2}{2m} + K\frac{x^2}{2}$ $\quad (K = m\omega^2)$

For $J = 0$, the eigenvalues are

$$\epsilon = \hbar\omega\left(n_1 + \tfrac{1}{2}\right) + \hbar\omega\left(n_2 + \tfrac{1}{2}\right).$$

The eigenfunctions of $H(x_1, x_2)$ are

$$\varphi = u_{n_1}(x_1)\, u_{n_2}(x_2)$$

n_1	n_2	ϵ
0	0	$\hbar\omega$
1	0	$2\hbar\omega$
0	1	$2\hbar\omega$
1	1	$3\hbar\omega$

The case of $J \neq 0$:

For ϵ_0, the first degree of perturbation energy is

$$\Delta\epsilon_0^{(1)} = J\langle\varphi_0|x_1 x_2|\varphi_0\rangle = J\langle u_0(x_1)|x_1|u_0(x_1)\rangle\langle u_0(x_2)|x_2|u_0(x_2)\rangle = 0$$

It doesn't change for the first.

So the second degree is.

$$\Delta\epsilon_0^{(2)} = \sum_k \frac{H'_{0k}H'_{k0}}{\epsilon_0 - \epsilon_k} = J^2\sum_k \frac{\langle\varphi_0|x_1x_2|\varphi_k\rangle\langle\varphi_k|x_1x_2|\varphi_0\rangle}{\epsilon_0 - \epsilon_k}$$

Using $\langle u_{n'}(x)|x|u_n(x)\rangle = \frac{1}{\alpha}\sqrt{n/2}\,\delta_{n',n-1} + \frac{1}{\alpha}\sqrt{(n+1)/2}\,\delta_{n',n+1}$ $\quad \left(\alpha = \sqrt{\frac{m\omega}{\hbar}}\right)$,

$$\langle u_0(x_1)|x_1|u_1(x_1)\rangle = \langle u_1(x_1)|x_1|u_0(x_1)\rangle = \frac{1}{\alpha}\sqrt{\frac{1}{2}}$$

Therefore,

$$\Delta\epsilon_0^{(2)} = \frac{J^2}{\hbar\omega - 3\hbar\omega}\left(\frac{1}{\alpha}\sqrt{\frac{1}{2}}\right)^4 = -\frac{J^2}{8\hbar\omega}\frac{1}{\alpha^4} = -\frac{\hbar}{8}\sqrt{\frac{K}{m}}\left(\frac{J}{K}\right)^2$$

The total energy is (ground state)

$$E_0 = \epsilon_0 + \Delta\epsilon_0^{(1)} + \Delta\epsilon_0^{(2)} = \hbar\sqrt{K/m}\,[1 - (J/K)^2/8]$$

For the degenerate,

① Find the matrix elements for $H' = Jx_1 x_2$.

The non-perturbe system $H(x_1, x_2)$ has 2 bases. degenerate eigenfunctions.

$$\varphi_{1,1} = u_1(x_1)\,u_0(x_2)\ , \qquad \varphi_{1,2} = u_0(x_1)\,u_1(x_2)$$

$$\begin{array}{c|cc} & \varphi_{11} & \varphi_{12} \\ \hline \varphi_{11} & 0 & \frac{J}{2\alpha^2} \\ \varphi_{12} & \frac{J}{2\alpha^2} & 0 \end{array} = H'$$

where

$$\langle \varphi_{11} | x_1 x_2 | \varphi_{11} \rangle = 0$$

$$\langle \varphi_{12} | x_1 x_2 | \varphi_{12} \rangle = \langle u_0(x_1) | x_1 | u_0(x_1) \rangle \langle u_1(x_2) | x_2 | u_1(x_2) \rangle$$

$$= 0$$

$$\langle \varphi_{11} | x_1 x_2 | \varphi_{12} \rangle = \langle u_1(x_1) | x_1 | u_0(x_1) \rangle \langle u_0(x_2) | x_2 | u_1(x_2) \rangle$$

$$= \frac{1}{\alpha}\sqrt{\frac{1}{2}} \cdot \frac{1}{\alpha}\sqrt{\frac{1}{2}}$$

$$= \frac{1}{2}\frac{1}{\alpha^2}$$

② Obtain ΔE .

$$\det|H' - \Delta E I| = \begin{vmatrix} -\Delta E & \frac{J}{2\alpha^2} \\ \frac{J}{2\alpha^2} & \cancel{\Delta} E \end{vmatrix} = 0$$

$$\det|H' - \Delta E I| = (\Delta E)^2 - \frac{J^2}{4\alpha^4} = 0$$

$$\therefore \quad \Delta E = \pm \frac{J}{2\alpha^2}$$

③ Find the total first excited state :

$$E_1 = E_1 + \Delta E = 2\hbar\omega \pm \frac{J}{2\alpha^2}$$