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Rotation Operator

- The eigenstate $|j, m = m_{max} = j\rangle$ is rotated about y-axis by an infinitesimal angle ϵ. Find the probability that the new rotated state can be found in the original state to the second order of ϵ.

We want to express J_y and J_y^2 in terms of ladder operators.

$$J_y = \frac{1}{2i}(J_+ - J_-)$$

$$J_y^2 = \frac{-1}{4}(J_+^2 + J_-^2 - J_+J_- - J_-J_+)$$

The expectation values for $|j, m=j\rangle$ are

$$\langle j, m=j | J_y | j, m=j\rangle = 0$$

$$\langle j, m=j | J_y^2 | j, m=j\rangle$$

$$= -\frac{1}{4}\left[\underbrace{\cancel{\langle j,m|J_+^2|j,m\rangle}}_{0} + \underbrace{\cancel{\langle j,m|J_-^2|j,m\rangle}}_{0} - \langle j,m|J_+J_-|j,m\rangle - \langle j,m|J_-J_+|j,m\rangle\right]$$

$$= \frac{1}{4}\left[\langle j,m|J_+J_-|j,m\rangle + \langle j,m|J_-J_+|j,m\rangle\right]$$

$$= \frac{1}{4}\langle j,m|J_+J_-|j,m\rangle \qquad (\because\ m = m_{max})$$

$$= \frac{1}{4}\langle j,m|J_-^\dagger J_-|j,m\rangle$$

$$= \frac{1}{4}(j+j)(j-j+1)\hbar^2$$

$$= \frac{1}{2}j\hbar^2$$

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The rotation operator is expressed as

$$\exp\left(\frac{iJ_y\epsilon}{\hbar}\right) \qquad \{\text{usually } \epsilon \text{ is } r_i\}$$

The matrix element of above for the state $|j, m=j\rangle$ is

$$d^{(j)}_{jj}(\epsilon) = \langle j, m=j|\exp\left(\frac{iJ_y\epsilon}{\hbar}\right)|j, m=j\rangle$$

$$= \langle j, m=j|1-\left(\frac{i}{\hbar}\right)J_y\epsilon + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 J_y^2\epsilon^2|j, m=j\rangle$$

$$= 1-\left(\frac{i}{\hbar}\right)\epsilon\times 0 + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2\epsilon^2\times\frac{1}{2}j\hbar^2$$

$$= 1-\frac{j}{4}\epsilon^2$$

The probability that we can find in the original state is:

$$P(\epsilon) = d^{(j)*}_{jj}(\epsilon)\, d^{(j)}_{jj}(\epsilon) = |d^{(j)}_{jj}(\epsilon)|^2 = \left(1-\frac{j}{4}\epsilon^2\right)^2$$

$$\approx 1-\frac{j}{2}\epsilon^2$$