Perturbation theory

- System

  1- dimensional S.H.O.

  perturbation term $Ax + Bx^3$

- Objective

  Find the second order perturbation.

Using

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^*)$$

$$a^*|n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$a|n\rangle = \sqrt{n}\,|n-1\rangle,$$

We can calculate

$$\boxed{Ax = A\sqrt{\frac{\hbar}{2m\omega}}\left\{\langle n|a|n\rangle + \langle n|a^*|n\rangle\right\}}$$

$$\boxed{\begin{aligned} Bx^3 &= B\left(\frac{\hbar}{2m\omega}\right)^{3/2}(aa + aa^* + a^*a + a^*a^*)(a + a^*) \\ &= B\left(\frac{\hbar}{2m\omega}\right)^{3/2}(aaa + aaa^* + aa^*a + aa^*a^* + a^*aa + a^*aa^* \\ &\qquad + a^*a^*a + a^*a^*a^*) \end{aligned}}$$

From the above, the first order perturbation is zero.

For the second order, we have to calculate.

$$\langle n+1|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left\{\sqrt{n+1}\,\langle n+1|n+1\rangle + \sqrt{n}\,\langle n+1|n-1\rangle\right\}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\sqrt{n+1}$$

$$\langle n-1|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left\{\sqrt{n+1}\,\langle n-1|n+1\rangle + \sqrt{n}\,\langle n-1|n-1\rangle\right\}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\sqrt{n}$$

$$\langle n+3|x^3|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{3/2}\left\{\sqrt{(n+1)(n+2)(n+3)}\,\langle n+3|n+3\rangle\right\}$$

$$\langle n-3|x^3|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{3/2}\left\{\sqrt{n(n-1)(n-2)}\,\langle n-3|n-3\rangle\right\}$$

$$\langle n+1|x^3|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{3/2}\left\{3(n+1)\sqrt{n+1}\,\langle n+1|n+1\rangle\right\}$$

$$\langle n-1|x^3|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{3/2}\left\{3n\sqrt{n}\,\langle n-1|n-1\rangle\right\}$$

The second order perturbation energy is

$$E_n^{(2)} = \frac{|\langle n-3|H'|n\rangle|^2}{\epsilon_n - \epsilon_{n-3}} + \frac{|\langle n-1|H'|n\rangle|^2}{\epsilon_n - \epsilon_{n-1}} + \frac{|\langle n+1|H'|n\rangle|^2}{\epsilon_n - \epsilon_{n+1}} + \frac{|\langle n+3|H'|n\rangle|^2}{\epsilon_n - \epsilon_{n+3}}$$

$$= \frac{B^2\left(\frac{\hbar}{2m\omega}\right)^3 n(n-1)(n-2)}{3\hbar\omega} + \frac{\left(A\sqrt{\frac{\hbar n}{2m\omega}} + B\left(\frac{\hbar}{2m\omega}\right)^{3/2} 3n\sqrt{n}\right)^2}{\hbar\omega} - \frac{\left(A\sqrt{\frac{\hbar}{2m\omega}}\sqrt{n+1} + B\left(\frac{\hbar}{2m\omega}\right)^{3/2} 3(n+1)\sqrt{n+1}\right)^2}{\hbar\omega}$$

$$- \frac{B^2\left(\frac{\hbar}{2m\omega}\right)^3 (n+1)(n+2)(n+3)}{3\hbar\omega}.$$