Question: There are two glasses that contain hot water (100 ^{o}C) and cold tea (0 ^{o}C), respectively. What if we increase the temperature of the tea by contacting with the hot water little by little using thermal equilibrium? (They will not mix together. Assume that heat does not go outside the system.)

1. Theory

In general, from conservation of thermal energy, the change of each heat must be equal. Heat is defined by

$Q = mcT$ where c is the heat capacity.

The change in this case is given by the difference of temperatures:

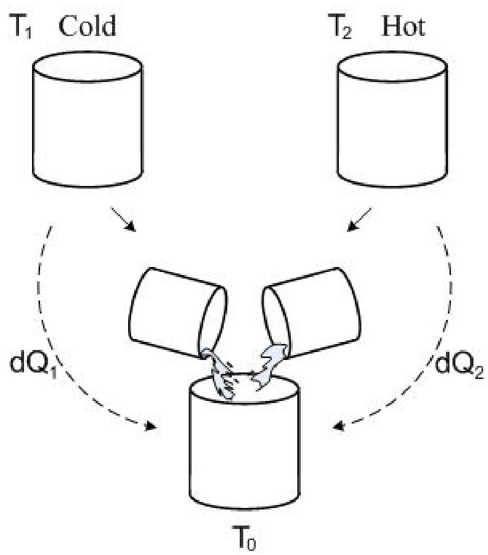
$$\Delta Q = mc\Delta T \qquad (1\text{-}1)$$

Therefore, the conservation of thermal energy is expressed as following:

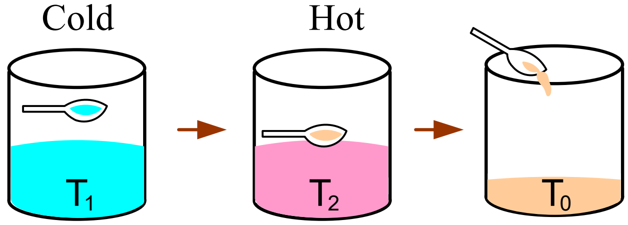
$$\Delta Q_1 = \Delta Q_2 \qquad (1\text{-}2)$$

Let T_0 be the equilibrium temperature. The above relationship can be rewritten as

$$m_1c_1(T_0 - T_1) = m_2c_2(T_2 - T_0) \qquad (1\text{-}3)$$



The question describes that the cold liquid is contacted with the hot one by each amount, m_1/N, and it is repeated N times to complete. The following figure may clarify this experiment:



We can assume that hot water and tea have the equal heat capacitance and mass. Thus, the equation becomes

$$\frac{mc}{N}(T_0 - T_1) = mc(T_2 - T_0) \qquad (1\text{-}4)$$

The *m* and *c* can be cancelled. In order to simplify, we let T_1 be 0 °C. Then, we have

$$\frac{1}{N}(T_0 - 0) = T_2 - T_0 \qquad (1\text{-}5)$$

Solve for T_0.

$$T_0 = \frac{T_2}{1+\frac{1}{N}} \qquad (1\text{-}6)$$

This T_0 is the equilibrium temperature of hot water after the first contact of a little spoon of cold tea. Then, we can plug T_0 into T_2 in equation (1-6). It gives

$$T_0 = \frac{T_2}{\left(1+\frac{1}{N}\right)^2} \qquad (1\text{-}7)$$

This can be repeated *N* times to complete the process. Thus, we have

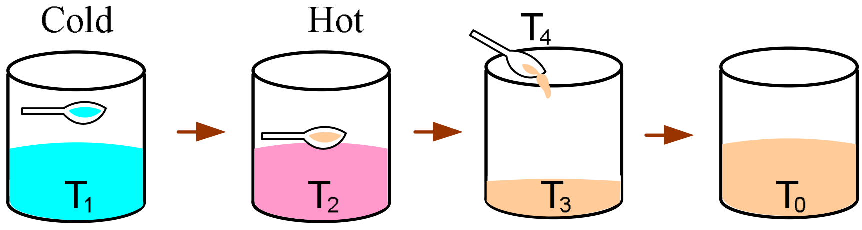
$$T_0 = \frac{T_2}{\left(1+\frac{1}{N}\right)^N} \qquad (1\text{-}8)$$

When *N* becomes infinity, $\left(1+\frac{1}{N}\right)^N$ becomes Napier's constant, $e = 2.7182$. If T_2 is 100 °C, the final temperature of the hot water will be

$$T_0 = \frac{100}{e} = 36.8\ ^oC \qquad (1\text{-}9)$$

This is the eventual temperature of hot reservoir.

What about the final temperature of cold tea? The following figure depicts the entire process:



The temperature, T_4, is the equilibrium temperature with T_1 and T_2, but T_2 decreases after each process. T_3 is the accumulated temperature in the third glass, which is mixed with T_4 and gives T_0. Thus, we can set up the following equation:

$$\frac{1}{N}\left(T_0 - T_{4n}\right) = \frac{n}{N}\left(T_3 - T_0\right) \qquad (1\text{-}10)$$

N and n are the total number of the process and current number of the process. T_{4n} is the current temperature of T_4. For the simplest case, we let T_1 be 0 °C. Namely,

$$T_{4n} = \frac{T_2}{\left(1+\frac{1}{N}\right)^n} \qquad (1\text{-}11)$$

The initial temperature of T_3 is derived from (1-11) when $n = 1$.

$$T_3^{\text{ini}} = \frac{T_2}{1+\frac{1}{N}} \qquad (1\text{-}12)$$

The final temperature, T_0, can be given from (1-10).

$$T_0 = \frac{nT_3 + T_{4n}}{1+n} \qquad (1\text{-}13)$$

This is replaced with the renewed T_3. After repeating many of the iterations, the final temperature will become 63.2 °C. (An algorithm of this general case is given in Appendix 2.) Notice that the cold tea becomes eventually more than 50 °C.

2. General extension

Now, let us generalize the process based on (1-5) for when T_1 not being equal to 0 °C. We will again assume that the heat capacity and mass are equal for both. The derivation can be done in the same manner; and k denotes the number of iteration:

$$k = 1: \ T_0 = \frac{T_2 + \frac{1}{N}T_1}{1+\frac{1}{N}} \qquad (2\text{-}1)$$

Let us define $M = 1 + \frac{1}{N}$. Then, the next several iterations will become as follows:

$$k = 2: \; T_0 = \frac{T_2 + \frac{T_1}{N}(1 + M)}{M^2} \tag{2-2}$$

$$k = 3: \; T_0 = \frac{T_2 + \frac{T_1}{N}(1 + M + M^2)}{M^3} \tag{2-3}$$

$$k = 4: \; T_0 = \frac{T_2 + \frac{T_1}{N}(1 + M + M^2 + M^3)}{M^4} \tag{2-4}$$

…

$$k = N: \; T_0 = \frac{T_2 + \frac{T_1}{N}(1 + M + M^2 \cdots + M^{N-1})}{M^N} \tag{2-5}$$

We can use $1 + M + M^2 \cdots + M^{N-1} = \frac{M^N - 1}{M - 1}$. Thus,

$$k = N: \; T_0 = \frac{T_2 + \frac{T_1}{N}\frac{M^N - 1}{M - 1}}{M^N} = \frac{T_2 + T_1(M^N - 1)}{M^N} \tag{2-6}$$

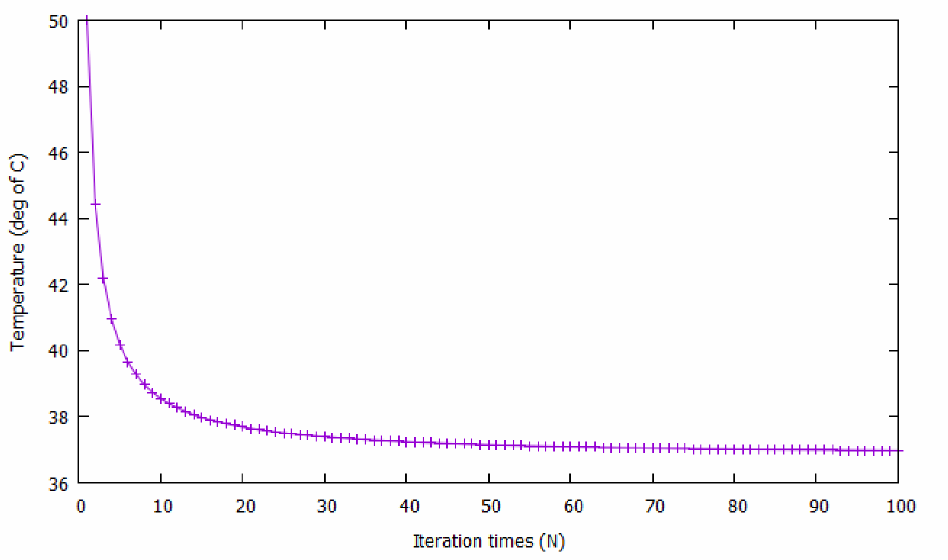
Since $\lim_{N \to \infty} M^N = e$, the limit of above expression is

$$T_0 = \frac{T_2 + T_1(e - 1)}{e} \tag{2-7}$$

(More general case including heat capacities and masses is derived in Appendix 1.)

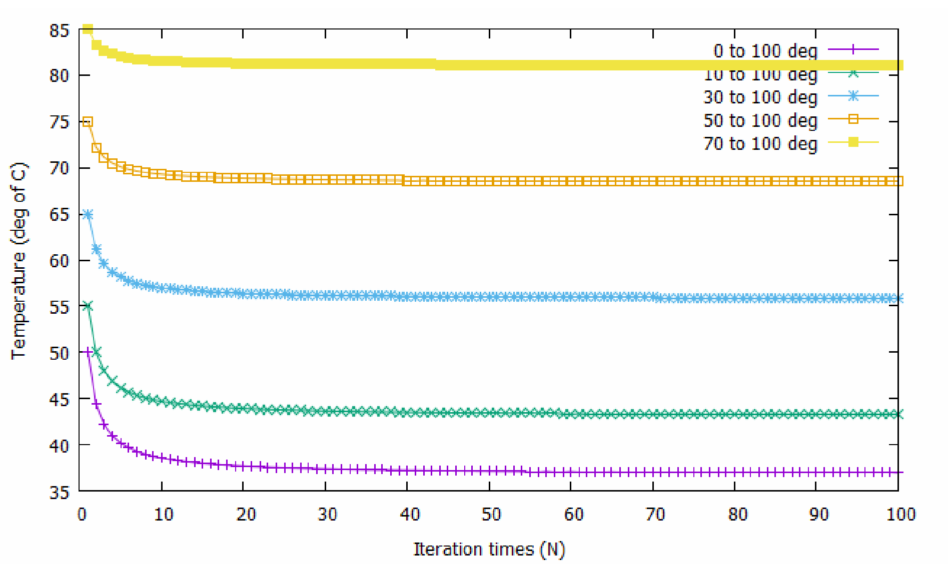
3. Numerical results

We can discuss this process from the following numerical results. This is the case when cold tea and hot water have 0 °C and 100 °C, respectively. Note that the following plots are the final temperature of hot reservoir.

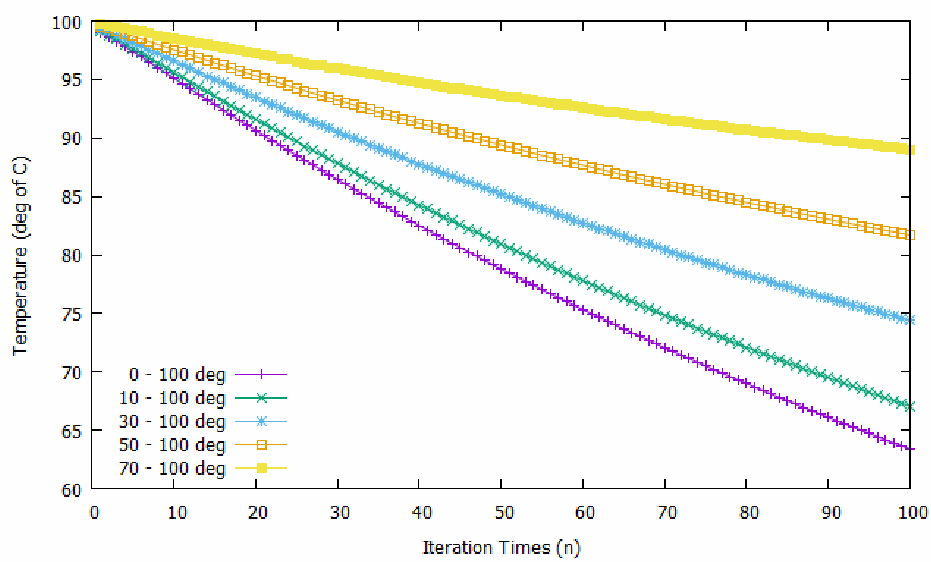


The x- and y-axes are the iteration process and the eventual temperature, respectively. You can see that one-time process gives 50 °C, but ten-time process yields 38.6 °C, which is 95 % of the possible limit temperature.

The following graph shows when T_1 has different temperatures. Each initial temperature is 0, 10, 30, 50, and 70 °C.



This figure shows the temperature change of tea for each initial temperature, 0, 10, 30, 50, and 70 °C.



Appendix 1

We can derive the formula for different heat capacities and masses. Let us start with the conservation of thermal energy.

$$\frac{m_1 c_1}{N}(T_1 - T_0) = m_2 c_2 (T_0 - T_2) \qquad \text{(A1-1)}$$

The process is exactly the same. Modify this equation as follows:

$$\frac{1}{N}(T_1 - T_0) = \frac{m_2 c_2}{m_1 c_1}(T_0 - T_2) \qquad \text{(A1-2)}$$

Define

$$C = \frac{m_2 c_2}{m_1 c_1} \qquad \text{(A1-3)}$$

Then, solve for T_0.

$$T_0 = \frac{C T_2 + \frac{1}{N} T_1}{\frac{1}{N} + C} \qquad \text{(A1-4)}$$

Put the denominator as follows:

$$M' = C + \frac{1}{N} \qquad \text{(A1-5)}$$

Substitute this into T_2 and repeat it 4 times.

$$T_0 = \frac{C^4 T_2 + \frac{T_1}{N}\left(C^3 + C^2 M' + C M'^2 + M'^3\right)}{M'^4} \qquad \text{(A1-6)}$$

We can generalize it until N iteration.

$$T_0 = \frac{C^N T_2 + \frac{T_1}{N}\sum_{i=1}^{N} C^{N-i} M'^{i-1}}{M'^N} \qquad \text{(A1-7)}$$

Appendix 2

The algorithm of the general case $(T_1 \neq 0)$ to obtain the final temperature of "cold" tea:

The initial temperature of T_3:

$$T_3^{\text{ini}} = \frac{T_2 + \frac{T_1}{N}}{1 + \frac{1}{N}} \qquad \text{(A2-1)}$$

Temperature of T_4 for each n (Refer to equation (2-6)):

$$T_{4n} = \frac{T_2 + T_1\left(M^N - 1\right)}{M^N} \qquad \text{(A2-2)}$$

The final temperature:

$$T_0 = \frac{nT_3 + T_{4n}}{1 + n} \qquad \text{(A2-3)}$$

Iterate this process:

$$T_3 \leftarrow T_0 \quad \text{and} \quad n \leftarrow n + 1 \qquad \text{(A2-4)}$$

This goes back to (A2-2), and will be repeated until N.

Reference

"Why Cats Land on Their Feet" by Mark Levi