Solving Newton's equations of motion

- Verlet method

The 2nd order ODE :

$$\frac{d^2x}{dt^2} = F(x,t) \qquad (1)$$

h as the discretized step. Let us expand x with Taylor expansion.

add
$$\begin{cases} x(h) = x(0) + h\dfrac{dx(0)}{dt} + \dfrac{h^2}{2!}\overbrace{\dfrac{d^2x(0)}{dt^2}}^{F(x,0)} + \dfrac{h^3}{3!}\dfrac{d^3x(0)}{dt^3} + O(h^4) & (2) \\ x(-h) = x(0) - h\dfrac{dx(0)}{dt} + \dfrac{h^2}{2!}\underbrace{\dfrac{d^2x(0)}{dt^2}}_{F(x,0)} - \dfrac{h^3}{3!}\dfrac{d^3x(0)}{dt^3} + O(h^4) & (3) \end{cases}$$

gives
$$x(h) = 2x(0) - x(-h) + h^2F(x,0) + O(h^4) \qquad (4)$$

This is the basic algorithm. However, we don't know $x(-h)$ in the first place, so the very first point has to be approximated by (3) :

$$x(-h) \approx x(0) - h\,v(0) + \frac{h^2}{2}F(x,0)$$

We can also have

$$v(h) \sim v(0) + \overbrace{\frac{F(h)+F(0)}{h}}^{\text{average}}\,h$$

This is called velocity Verlet.

Leapfrog (Stömer) Method

For x, $\quad x(h) \sim x(0) + v(0)\cdot h + \frac{1}{2} F(0) h^2$

For v, $\quad v(h) \sim v(0) + \frac{F(h) + F(0)}{2} h \quad$ (same as Verlet)