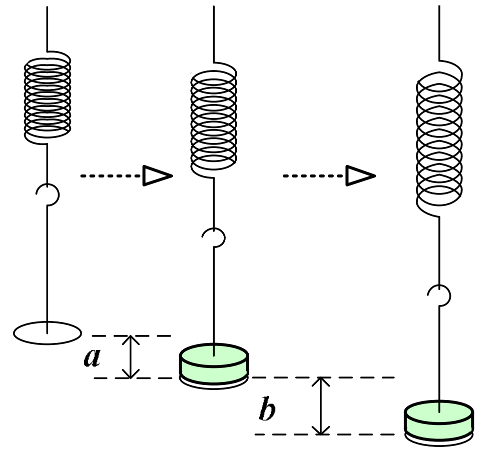
The vertical spring motion



Before placing a mass on the spring, it is recognized as its natural length. The spring constant is k, and the displacement of a will be given as follows:

$$|F| = ka = mg \Rightarrow a = \frac{mg}{k}$$

The Newton's equation of motion from the equilibrium point by stretching an extra length as shown is:

$$\sum F = mg - k(a+b) = ma$$

$$\begin{aligned} ma &= mg - k\left(\frac{mg}{k} + b\right) \\ &= mg - mg - kb \\ &= -kb \end{aligned}$$

Note that this equation of motion does not contain the gravitational force after all. The gravitational force is canceled and included as an updated restoring force of the spring.

Now, consider the motion with conservation of energy. The total energies for initial and final are expressed as

$$\begin{aligned} \frac{1}{2}mv^2 + \frac{1}{2}k(a+b)^2 - mgb &= \frac{1}{2}mv^2 + \frac{1}{2}k(a^2 + 2ab + b^2) - mgb \\ &= \frac{1}{2}mv^2 + \frac{1}{2}ka^2 + kab + \frac{1}{2}ka^2 - mgb \end{aligned}$$

Since $ka = mg$, we can manipulate it as $kab = mgb$. Therefore, the above two terms cancel out. Then we have following:

The total energy = $\frac{1}{2}mv^2 + \frac{1}{2}kb^2 + \frac{1}{2}ka^2$

The term $\frac{1}{2}ka^2$ is at the equilibrium, and it can be physically neglected from the total energy. Thus, we have the final relationship:

The total energy = $\frac{1}{2}mv^2 + \frac{1}{2}kb^2$, which is simply the combination of the kinetic energy and spring potential energy.