Fundamental Optical Interaction Theory

The main reference is "Optical Resonance and Two-level Atoms" by L. Allen and J. H. Eberly.

1. <u>Classical Theory of Resonance Optics</u>

1.1 Introduction

Classically, H. A. Lorentz created the theory of interaction between light and matters. The research tells that the phenomena can be described in terms of dipole moment and electromagnetic fields. In other words, the optical properties of matter depend on the dielectrics. Before going to the quantum theory, we derive the familiar results.

1.2 The linear dipole oscillator

The optical phenomena can be interpreted as an interaction between electromagnetic field and electric charges. If we assume that these charges are bounded as neutral atoms, there are electron-ion pairs behaving as simple harmonic oscillators. Namely, the entire system is composed with many harmonic oscillators and the electric dipole moments. That is:

$$\mathrm{H} = \frac{1}{2m}\sum_a \left(p_a^2 + \omega_a^2 m^2 r_a^2\right) - e\sum_a \mathbf{r}_a \cdot \mathbf{E}\left(t, \mathbf{r}_a\right) \tag{1.1}$$

where ω_a is the natural oscillation frequency. From the Hamilton (1.1), we have the classical equation of motion.

$$\ddot{x}_a + \omega_a^2 x_a = \frac{e}{m} E\left(t, \mathbf{r}_a\right) \tag{1.2}$$

As known, Eq. (1.2) is derived by two Hamilton Eqs. This is called a forced linear oscillation. However, we have to take the radiation by itself into account, which is known as energy decay. The following is the continuity formula in terms of the Poynting vector:

$$\nabla \cdot S + \frac{\partial U_{mat}}{\partial t} = 0 \tag{1.3}$$

where U_{mat} represents the energy density in a material. We omit the electromagnetic energy (induced by the fields) because it is very small compared with the electron dipole oscillation. If we integrate (1.3) over a sphere and use Gauss' law, we obtain

$$\int \mathbf{S}\cdot\mathbf{n}\,dA + \frac{\partial W_{osc}}{\partial t} = 0 \tag{1.4}$$

We can approximate the potential energy as

$$W_a(t) = m\omega_a^2 x_a^2(t). \tag{1.5}$$

Purposely, we took $W_a(t) = m\omega_a^2 x_a^2(t)$ instead of $W_a(t) = \frac{1}{2} m\omega_a^2 x_a^2(t)$, and the dipole distance x is averaged. Now we can consider the elementary dipole-radiation problem. The Poynting vector is

$$S = \frac{p^2\omega^4}{4\pi c^3 R^2}\sin^2\theta \qquad \text{\{Gaussian Unit\}} \tag{1.6}$$

[Refer to Born and Wolf *Principles of Optics* Section 2.2.3, or Vanderlinde *Classical Electromagnetic Theory* Section 10.2.1] The rate of energy loss by electric dipole radiation is known as follows:

$$\begin{aligned}\int S\,d\sigma &= \frac{p^2\omega^4}{4\pi c^3 R^2}\int_0^{\pi}\sin^2\theta\cdot R^2\sin\theta\,d\theta\int_0^{2\pi} d\phi \\ &= \frac{2p^2\omega^4}{3c^3} \\ &= \frac{2e^2\omega_a^4}{3c^3}x_a^2(t)\end{aligned} \tag{1.7}$$

It is radiated over a sphere and the dipole is placed at the center. From (1.5) and (1.7), we obtain

$$\int S\,d\sigma = \frac{2e^2\omega_a^2}{3mc^3}W_a(t). \tag{1.8}$$

Using (1.4), we have the following equation:

$$\frac{\partial W_a}{\partial t} = -\frac{2}{\tau_0} W_a \tag{1.9}$$

where $\frac{2}{\tau_0} = \frac{2e^2\omega_a^2}{3mc^3}$. Solving (1.9) for W_a, we obtain

$$W_a(t) = W_a(0)\exp[-2t/\tau_0] \tag{1.10}$$

where $W_a(0)$ is the initial value at $t = 0$. The formula, (1.10), expresses the oscillation decay. If the angular frequency ω_a is approximately $10^{15} \sim 10^{20}$ Hz (optical region), the natural lifetime will become the order of 0.1×10^{-6} sec; namely, $1/\tau_0 << \omega_a$. Thus, we can insert the decay term into (1.2):

$$\ddot{x}_a + \frac{2}{\tau_0}\dot{x}_a + \omega_a^2 x_a = \frac{e}{m}E(t,\mathbf{r}_a) \tag{1.11}$$

For $\frac{2}{\tau_0} << \omega_a$, the decay rate is slow. Although the term $\frac{2}{\tau_0}$ is small, the rate is still fast.

1.3 The Classical Rabi Problem

We consider a resonance effect of equation (1.11). The solution for (1.11) can be obtained straightforwardly with an algebraic process. The driving field can be described as $E = \varepsilon(e^{i\omega t} + c.c.)$. Then we assume the special solution of the differential equation, (1.11), as $x_a = Be^{i\omega t}$. We know the solution as follows:

$$B = \frac{e\varepsilon/m}{\omega_a^2 - \omega^2 + \frac{2i}{\tau_0}\omega}. \tag{1.12}$$

Employing the complex variables notation, $B = be^{i\delta}$, we have

$$b = \frac{e\varepsilon/m}{\sqrt{\left(\omega_a^2 - \omega^2\right)^2 + \frac{4}{\tau_0^2}\omega^2}}, \qquad \tan\delta = \frac{2\omega/\tau_0}{\omega^2 - \omega_a^2} \tag{1.13}$$

[Refer to Landau, and Lifshits *Classical Mechanics* Section 26]. Next, we assume a general solution for (1.11) as follows:

$$x_a = x_0\left[u_a \cos\omega t - v_a \sin\omega t\right]. \tag{1.14}$$

A constant, x_0, can be regarded as the amplitude of the oscillation at some arbitrary time. The variables, u_a and v_a, are called the envelope functions. They are assumed to be varied very slowly when the difference $\omega - \omega_a$ is very small. Now we take the first and second derivative of (1.14) with respect to time:

$$\dot{x}_a = x_0\left[(\dot{u}_a - \omega v_a)\cos\omega t - (\omega u_a + \dot{v}_a)\sin\omega t\right] \tag{1.15}$$

$$\ddot{x}_a = x_0\left[\left(\ddot{u}_a - \omega^2 u_a - 2\omega\dot{v}_a\right)\cos\omega t + \left(-\ddot{v}_a + \omega^2 v_a - 2\omega\dot{u}_a\right)\sin\omega t\right] \tag{1.16}$$

Plug them into (1.11) and recall $E = \frac{e}{m}\varepsilon\left(e^{i\omega t} + c.c.\right)$.

$$\begin{aligned} &\left\{\ddot{u}_a + \left(\omega_a^2 - \omega^2\right)u_a - 2\omega\dot{v}_a + \frac{2}{\tau_0}(\dot{u}_a - \omega v_a)\right\}\cos\omega t \\ &+ \left\{-\ddot{v}_a - \left(\omega_a^2 - \omega^2\right)v_a - 2\omega\dot{u}_a - \frac{2}{\tau_0}(\omega u_a + \dot{v}_a)\right\}\sin\omega t = \frac{e}{x_0 m}\varepsilon\cos\omega t \end{aligned} \tag{1.17}$$

Remember the assumption that the envelope functions are changing slowly compared with frequency of the driving field. In other expressions, we have

$$\dot{u}_a << \omega u_a, \quad \ddot{u}_a << \omega^2 u_a, \quad \dot{v}_a << \omega v_a, \quad \ddot{v}_a << \omega^2 v_a \tag{1.18}$$

We can neglect the contribution from the second derivative of u_a and v_a. Comparing the both sides of the equation, we obtain a pair of equations for u_a and v_a:

$$\dot{u}_a = -\frac{1}{2\omega}\left(\omega_a^2 - \omega^2\right)v_a - \frac{u_a}{\tau_0} - \frac{1}{\omega\tau_0}\dot{v}_a, \tag{1.19}$$

$$\dot{v}_a = \frac{1}{2\omega}\left(\omega_a^2 - \omega^2\right)u_a - \frac{v_a}{\tau_0} - \left(\frac{e}{m\omega x_0}\right)\varepsilon + \frac{1}{\omega\tau_0}\dot{u}_a \tag{1.20}$$

Let us go back to the discussion on resonance when $\omega_a \approx \omega$. We can approximate $\omega_a^2 - \omega^2$ as follows:

$$\begin{aligned}\omega_a^2 - \omega^2 &= (\omega_a + \omega)(\omega_a - \omega) \\ &= \omega\left(\tfrac{\omega_a}{\omega} + 1\right)(\omega_a - \omega) \\ &\approx 2\omega(\omega_a - \omega).\end{aligned} \tag{1.21}$$

Our earlier discussion for the radiative decay is relatively slow; namely, $\omega\tau_0 >> 1$. Using this condition and (1.21), we can eliminate the last term of them and we rewrite (1.19) and (1.20) as

$$\dot{u} = -\Delta v - \frac{u}{T} \tag{1.22a}$$

$$\dot{v} = \Delta u - \frac{v}{T} + \kappa\varepsilon \tag{1.22b}$$

where $\Delta = \omega_a - \omega$ and $\kappa =: \frac{e}{m\omega x_0}$. The reason why we use T instead of τ_0 is that the random interaction makes the life time shorter. (It depends on the circumstance.) The symbols T and τ_0 are called effective radiative lifetime and purely radiative lifetime, respectively. The decay rate is $T^{-1} \geq \tau_0$. The above differential equations give us the following solutions:

$$u(t:\Delta) = [u_0 \cos\Delta t - v_0 \sin\Delta t]e^{-t/T} + \kappa\varepsilon\int_0^t dt' \sin\Delta(t-t')e^{-(t-t')/T} \tag{1.23a}$$

$$v(t:\Delta) = [u_0 \sin\Delta t + v_0 \cos\Delta t]e^{-t/T} - \kappa\varepsilon\int_0^t dt' \cos\Delta(t-t')e^{-(t-t')/T} \tag{1.23b}$$

After a long time, all initial oscillations will be vanished. Therefore, we have the results shown as (1.12) and (1.13). In other expressions,

$$x_a(t) = \frac{e}{m}\varepsilon\left(\frac{e^{i\omega t}}{\omega_a^2 - \omega^2 + 2i\omega/T} + c.c.\right). \tag{1.24}$$

Summary of Chap. 1

- An optical interaction between driving field and electric charges can be expressed as a forced linear oscillation.
- The solution (1.14) described in terms of envelope functions leads to significant result (1.23).
- The solutions (1.23) are useful for the short time analysis and the resonant case.