3. Two-Level Atoms in Steady fields

3.1 Introduction

We here deal with a class of interactions in which the amplitude of electric fields does not change with time. The equations derived in section 2.4 can be solved by Rabi's method, which uses the oscillations as analogues of those in magnetic resonance.

3.2 π Pulses

The equations derived in section 2.4 are fundamental since these can be applied to more complex interactions. The Rabi solution is the simplest for the cases exactly at resonance with the laser field. Now, let us solve the differential equations, (2.27a). The following notation is be defined as

$$\theta(t) = \int_{-\infty}^{t} \kappa\varepsilon(t')dt' . \tag{3.1}$$

The equations are usually solved for v and w with the conditions that $u(t;0) = u(0;0) = u_0$. Remember equations (2.27):

$$\dot{u} = -\Delta v \tag{3.2}$$

$$\dot{v} = \Delta u + \kappa\varepsilon w \tag{3.3}$$

$$\dot{w} = -\kappa\varepsilon v \tag{3.4}$$

where $\Delta = \omega_0 - \omega$. From (3.3), we have

$$\ddot{v} = \Delta\dot{u} + \kappa\varepsilon\dot{w} \tag{3.5}$$

Using (3.2) and (3.4), we obtain

$$\ddot{v} + \left(\Delta^2 + \kappa^2\varepsilon^2\right)v = 0 \tag{3.6}$$

Solve (3.6).

$$v = C_1 e^{i\varpi t} + C_2 e^{-i\varpi t} \tag{3.7}$$

where C_1 and C_2 are constants, and $\varpi = \Delta + \kappa\varepsilon$. Thus,

$$v = A\cos\varpi t + B\sin\varpi t \tag{3.8}$$

where A and B are some constants. Using trigonometric addition formulae, we obtain

$$v = \left(A\cos\Delta t + B\sin\Delta t\right)\cos\kappa\varepsilon t + \left(-A\sin\Delta t + B\cos\Delta t\right)\sin\kappa\varepsilon t . \tag{3.9}$$

We can rewrite it as

$$v(t;0) = w_0 \sin\theta(t) + v_0 \cos\theta(t). \tag{3.10}$$

Put (3.10) into (3.4).

$$w(t;0) = -v_0 \sin\theta(t) + w_0 \cos\theta(t) \tag{3.11}$$

Consequently we have the solutions as follows:

$$\begin{cases} u(t;0) = u_0 \\ v(t;0) = w_0 \sin\theta(t) + v_0 \cos\theta(t) \\ w(t;0) = -v_0 \sin\theta(t) + w_0 \cos\theta(t) \end{cases} \tag{3.12}$$

In the case in which the applied field envelope has the steady value ε_0 from t_1 to t_2, equation (3.1) can be integrated to give:

$$\theta = \kappa\varepsilon_0 (t_2 - t_1) = \Omega(0)(t_2 - t_1), \tag{3.13}$$

where Ω is called the Rabi frequency on resonance. The Rabi frequency gives the rate at which transitions are coherently induced between the two atomic levels. It should be noted the following property of the solution (3.12): When $\theta(t) = \kappa\varepsilon_0 \delta t \equiv \pi$, which is called π pulse, it will exactly invert an atom at the ground-state. In fact, it makes sense if we substitute $\theta = 0$ and $\theta = \pi$. π pulse can actually be expressed as follows:

$$A(t) = \kappa \int_{-\infty}^{t} \varepsilon(t')dt' = \theta(t). \tag{3.14}$$

Eq. (3.14) implies that the π pulse is the area of the envelope wave. Therefore, a Rabi frequency determines the rate of the duration. The area $\theta = n\pi \ (n = 0,1,2\ldots)$ can be called resonant pulses. The total inversion of the Bloch vector $\vec{\rho}(t;0)$ is equivalent with resonance. There is another important property. Equations (2.27a) express motion in the rotating frame. We can rewrite it as

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & -\Delta & 0 \\ \Delta & 0 & \kappa\varepsilon \\ 0 & -\kappa\varepsilon & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{2.27a}$$

When Δ is zero, the above will be the precession about the **1** axis only.

3.3 The Rabi Solution

In general, it is difficult to solve equation (2.27b), but for Rabi case in which ε is steady with respect to time can have a solution. We here discuss the derivation of the solution from successive rotation transformations. In the off-resonance Rabi case, the torque vector $\boldsymbol{\Omega} = \left(-\kappa\varepsilon, 0, \Delta\right)$ [Refer to (2.27c).] in the rotating frame is a constant vector on the $\mathbf{1}-\mathbf{3}$ plane. The angle is defined as

$$\chi = \tan^{-1}\frac{\Delta}{\kappa\varepsilon_0}, \qquad \text{or} \quad \tan\chi = \frac{\Delta}{\kappa\varepsilon_0}. \tag{3.15}$$

This rotation projects $\boldsymbol{\Omega}$ onto the $-\mathbf{1}'$ axis. The Bloch vector, ρ, is also influenced by the coordinate rotation, which becomes $\rho' = \left(u', v', w'\right)$. The rotation is explicitly given as

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos\chi & 0 & \sin\chi \\ 0 & 1 & 0 \\ -\sin\chi & 0 & \cos\chi \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}. \tag{3.16}$$

This is replaced as follows:

$$\vec{\rho} = \mathbf{B}\vec{\rho}'. \tag{3.16)'$$

According to (2.27a), we have

$$\frac{d}{dt}\vec{\rho} = \mathbf{A}\vec{\rho} \tag{3.17}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -\Delta & 0 \\ \Delta & 0 & \kappa\varepsilon \\ 0 & -\kappa\varepsilon & 0 \end{bmatrix}. \tag{3.18}$$

From (3.16)’ and (3.17), we get

$$\frac{d}{dt}\mathbf{B}\vec{\rho}' = \mathbf{A}\mathbf{B}\vec{\rho}'. \tag{3.19}$$

Therefore,

$$\dot{\vec{\rho}}' = \left[\mathbf{B}^{-1}\mathbf{A}\mathbf{B} - \mathbf{B}^{-1}\dot{\mathbf{B}}\right]\vec{\rho}'. \tag{3.20}$$

However, $\mathbf{B}$ is constant with respect to time. Thus,

$$\dot{\vec{\rho}}' = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}\vec{\rho}'. \tag{3.21}$$

We can calculate it easily.

$$\frac{d}{dt}\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 0 & -\Delta\cos\chi + \kappa\varepsilon_0 \sin\chi & 0 \\ \Delta\cos\chi - \kappa\varepsilon_0 \sin\chi & 0 & \Delta\sin\chi + \kappa\varepsilon_0\cos\chi \\ 0 & -\Delta\sin\chi - \kappa\varepsilon_0\cos\chi & 0 \end{bmatrix}\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} \quad (3.22)$$

where

$$\cos\chi = 1\Big/\sqrt{1+\tfrac{\Delta^2}{(\kappa\varepsilon_0)^2}}\,, \qquad \sin\chi = \Delta\Big/\kappa\varepsilon_0\sqrt{1+\tfrac{\Delta^2}{(\kappa\varepsilon_0)^2}}\,,$$

according to (3.16). Eq. (3.22) is rewritten as

$$\frac{d}{dt}\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega(\Delta) \\ 0 & \Omega(\Delta) & 0 \end{bmatrix}\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}. \quad (3.22)'$$

We can see the pseudospin $\vec{\rho}\,'$ is now precessing about the new $-\mathbf{1}$ axis with the following frequency:

$$\Omega(\Delta) = \sqrt{\Delta^2 + (\kappa\varepsilon_0)^2}\,. \quad (3.23)$$

The frequency (3.23) is the Rabi frequency generalized for detuning. The counter-rotation about the $\mathbf{1}'$ axis through the angle $-\Omega(\Delta)t$ will lead to a coordinate frame in which the pseudospin vector $\vec{\rho}\,''$ is stationary, where it is given as follows:

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega t & \sin\Omega t \\ 0 & -\sin\Omega t & \cos\Omega t \end{bmatrix}\begin{bmatrix} u'' \\ v'' \\ w'' \end{bmatrix}. \quad (3.24)$$

At $t = 0$, we assume that

$$u_0 \equiv u(0;\Delta), \quad v_0 \equiv v(0;\Delta), \quad w_0 \equiv w(0;\Delta). \quad (3.25)$$

So we can relate $\vec{\rho}\,''$ to $\vec{\rho}_0$ as

$$\begin{bmatrix} u'' \\ v'' \\ w'' \end{bmatrix} = \begin{bmatrix} \cos\chi & 0 & -\sin\chi \\ 0 & 1 & 0 \\ \sin\chi & 0 & \cos\chi \end{bmatrix}\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}. \quad (3.26)$$

As a result,

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos\chi & 0 & \sin\chi \\ 0 & 1 & 0 \\ -\sin\chi & 0 & \cos\chi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega t & \sin\Omega t \\ 0 & -\sin\Omega t & \cos\Omega t \end{bmatrix} \begin{bmatrix} \cos\chi & 0 & -\sin\chi \\ 0 & 1 & 0 \\ \sin\chi & 0 & \cos\chi \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}. \qquad (3.27)$$

Thus,

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{(\kappa\varepsilon_0)^2 + \Delta^2 \cos\Omega t}{\Omega^2} & \frac{-\Delta}{\Omega}\sin\Omega t & \frac{-\Delta\kappa\varepsilon_0}{\Omega^2}(1-\cos\Omega t) \\ \frac{\Delta}{\Omega}\sin\Omega t & \cos\Omega t & \frac{\kappa\varepsilon_0}{\Omega}\sin\Omega t \\ \frac{-\Delta\kappa\varepsilon_0}{\Omega^2}(1-\cos\Omega t) & \frac{-\kappa\varepsilon_0}{\Omega}\sin\Omega t & \frac{\Delta^2 + (\kappa\varepsilon_0)^2\cos\Omega t}{\Omega^2} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}. \qquad (3.28)$$

We can easily find the limit of zero detuning as $\Delta \to 0$. Expression (3.28) results in

$$\begin{cases} u(t;0) = u_0 \\ v(t;0) = w_0 \sin\theta(t) + v_0 \cos\theta(t) \\ w(t;0) = -v_0 \sin\theta(t) + w_0 \cos\theta(t) \end{cases} . \qquad (3.12)$$

Consider the initial condition $u_0 = v_0 = 0$ and $w_0 = -1$ for w:

$$w(t;\Delta) = -1 + \frac{2(\kappa\varepsilon_0)^2}{(\kappa\varepsilon_0)^2 + \Delta^2} \sin^2 \sqrt{(\kappa\varepsilon_0)^2 + \Delta^2}\, \frac{t}{2}. \qquad (3.29)$$

Note that if the system is tuned closely enough to exact resonance so that $\Delta < \kappa\varepsilon_0$, then appreciable inversion occurs with each cycle.

Summary of Chap. 3

- The solution for (3.2-4) indicates the inversion property, which is called π pulse.
- The system in steady field is called Rabi case.
- Rabi solution is derived from coordinate transformations and the classical concept of the motion.