What is effective range theory?[1]

The reason why this theory has to be used is that the neutron-proton scattering cross-section, σ_{np} becomes too large at low energy. If the incident energy is small enough, the S-wave ($l = 0$) only becomes effective. Let us start with the Schrödinger equation:

$$\left\{ \frac{d^2}{dr^2} + k^2 - U(r) \right\} u_k(r) = 0 \tag{1}$$

where $k^2 = \frac{2\mu E}{\hbar^2}$ and $U(r) = \frac{2\mu}{\hbar^2} V(r)$. The boundary conditions of $u_k(r)$ are given in terms of phase shift, δ:

$$u_k(0) = 0; \qquad u_k(r \to \infty) \longrightarrow \frac{\sin(kr + \delta)}{\sin \delta} \tag{2}$$

The wave function, $u_0(r)$, is the one when the incident energy is zero, $k = 0$. The equation is given as follows:

$$\left\{ \frac{d^2}{dr^2} - U(r) \right\} u_k(r) = 0 \tag{3}$$

From (1) and (3), eliminate the potential, $U(r)$.

$$u_0 \frac{d^2 u_k}{dr^2} - u_k \frac{d^2 u_0}{dr^2} = -k^2 u_0 u_k \tag{4}$$

The above equation holds even when $U(r) = 0$. Thus we can have the same discussion with the following Schrödinger equation with a different wave function:

$$\left\{ \frac{d^2}{dr^2} + k^2 \right\} w_k(r) = 0 \tag{5}$$

aloong with the boundary condition:

$$w_k(r \to \infty) \longrightarrow u_k(r \to \infty) \tag{6}$$

This also holds the same equation as (4). Namely,

$$w_0 \frac{d^2 w_k}{dr^2} - w_k \frac{d^2 w_0}{dr^2} = -k^2 w_0 w_k \tag{7}$$

Take the difference between equations (4) and (7) and integrate it from 0 to ∞.

$$[u_0 u_k' - u_k u_0']_0^\infty - [w_0 w_k' - w_k w_0']_0^\infty = k^2 \int_0^\infty (w_0 w_k - u_0 u_k) dr \tag{8}$$

Note that the prime denotes the derivative in terms of r. According tothe boundary conditions, $[u_0 u_k' - u_k u_0']_0^\infty$ will not survive. Only when $r = 0$, we can have the following:

$$\left[w_0 \frac{dw_k}{dr} - w_k \frac{dw_0}{dr} \right]_{r=0} = k^2 \int_0^\infty (w_0 w_k - u_0 u_k) dr \tag{9}$$

Remember the boundary condition, (6):

$$\begin{aligned} w_k(r) &= \frac{\sin(kr + \delta)}{\sin \delta} \\ &= \frac{\sin kr \cos \delta + \sin \delta \cos kr}{\sin \delta} \\ &= \cos kr + \cot \delta \sin kr \end{aligned} \tag{10}$$

From this, we can derive $w_0(r)$ which is the asymptotic function when $k \to 0$:

$$\begin{aligned} w_0(r) &\equiv \lim_{k \to 0} w_k(r) \\ &= \lim_{k \to 0} \cos kr + \lim_{k \to 0} \frac{kr \sin kr}{kr} \cot \delta \\ &= 1 + r \lim_{k \to 0} k \cot \delta \end{aligned} \tag{11}$$

From the related dimension, we can define the scattering length from above:

$$\frac{1}{a} \equiv - \lim_{k \to 0} k \cot \delta \tag{12}$$

Thus, the equation (11) can be written as

$$w_0(r) = 1 - \frac{r}{a} \tag{13}$$

We now have w_0 and w_k in equations (10) and (13). These will be substitute in the left-hand side of equation (9):

$$\begin{aligned} & \left(1 - \frac{r}{a}\right) \frac{d}{dr} (\cos kr + \cot \delta \sin kr) - (\cos kr + \cot \delta \sin kr) \frac{d}{dr} \left(1 - \frac{r}{a}\right)_{r=0} \\ = & \left(1 - \frac{r}{a}\right) (-k \cos kr + k \cot \delta \cos kr) - (\cos kr + \cot \delta \sin kr) \left(-\frac{1}{a}\right)_{r=0} \\ = & \; k \cot \delta + \frac{1}{a} \end{aligned} \tag{14}$$

Equate this with the right-hand-side of (9).

$$k \cot \delta = -\frac{1}{a} + k^2 \int_0^\infty (w_k w_0 - u_0 u_k) dr \tag{15}$$

The second term of the right-hand-side can be expanded with k^2; then, we have

$$k \cot \delta = -\frac{1}{a} + k^2 r_0 + O(k^4) \tag{16}$$

where

$$r_0 \equiv 2 \int_0^\infty (w_0^2 - u_0^2) dr \tag{17}$$

and this is called effective range. Equation (16) is known as the effective range formula. The phase shift at low energy is given by the scattering length and effective range. For $k \to 0$, the formula gives

$$\begin{aligned} k \cot \delta &\sim -\frac{1}{a} \\ \Longrightarrow \tan \delta &= -ka \end{aligned} \tag{18}$$

The cross-section is given by

$$\sigma = \frac{4\pi}{k^2} \sin^2 \delta \tag{19}$$

When δ is very small, $\sin \delta \sim \tan \delta$. Thus,

$$\sigma \sim \frac{4\pi}{k^2} (-ka)^2 = 4\pi a \tag{20}$$

This result makes sense when the incident energy is close to zero.

References

[1] J. J. Sakurai, *Advanced Quantum Mechanics*. Addison-Wesley,1967