# Legendre transformation for the Lagrangian of non-linear $\sigma$ model[1]

The Lagrangian of non-linear sigma model reads

$$\mathcal{L} = -\frac{1}{2}\sum_{mn} \partial_\mu \Phi^n \partial^\mu \Phi^m f_{nm}(\Phi)$$

Note that $\partial_\mu \Phi \to \dot{\Phi}$. The generalized momentum is derived:

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = -\sum_{mn} \dot{\Phi}^{mn} f_{nm}(\Phi)$$

Therefore, we can solve for $\sum \dot{\Phi}$:

$$\sum_{mn} \dot{\Phi}^{mn} = -\frac{\Pi}{\sum_{mn} f_{nm}(\Phi)}$$

Then, according to Legendre transformation, we derive the Hamiltonian.

$$\begin{aligned}
\mathcal{H} &= \int d^3x [\Pi \sum \dot{\Phi} - \mathcal{L}] \\
&= \int d^3x \left[\Pi \left(-\frac{\Pi}{\sum f(\Phi)}\right) + \frac{1}{2}\left(-\frac{\Pi}{\sum f(\Phi)}\right)^2 \sum f_{nm}(\Phi)\right] \\
&= \int d^3x \left[-\frac{\Pi^2}{\sum f(\Phi)} - \frac{1}{2}\frac{\Pi^2}{\sum f(\Phi)}\right] \\
&= \int d^3x \left[-\frac{3}{2}\frac{\Pi^2}{\sum f(\Phi)}\right]
\end{aligned}$$

## References


[1] S. Weinberg, *The Quantum Theory of Fields.* Addison-Wesley, Reading MA, 1986