The field theory with basic derivations[1]

The scalar field

The Lagrangian reads

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\partial \phi}{\partial x_\mu}\frac{\partial \phi}{\partial x_\mu} + \mu^2\phi^2\right)$$

Plug it in the following Euler-Lagrange equation:

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x_\mu)}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

We have

$$\to \Box\phi - \mu^2\phi = 0$$

$$\to \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi - \mu^2\phi = 0$$

In quantum mechanics,, $E \to i\hbar\frac{\partial}{\partial t}$ and $p_k \to -i\hbar\frac{\partial}{\partial x_k}$. Therefore,

$$\nabla = \frac{iP}{\hbar} \implies \nabla^2 = -\frac{P^2}{\hbar^2}$$
$$\frac{\partial}{\partial t} = \frac{E}{i\hbar} \implies \frac{\partial^2}{\partial t^2} = -\frac{E^2}{\hbar^2}$$

Thus,

$$-\frac{P^2}{\hbar^2}\phi + \frac{1}{c^2}\frac{E^2}{\hbar^2}\phi = \mu^2\phi$$

Since $E^2 = m^2c^4 + p^2c^2$, it will be

$$-\frac{P^2}{\hbar^2}\phi + \frac{1}{c^2}\frac{m^2c^4 + p^2c^2}{\hbar^2}\phi = \mu^2\phi$$

$$-\frac{P^2}{\hbar^2}\phi + \frac{m^2c^2 + p^2}{\hbar^2}\phi = \mu^2\phi$$

$$\frac{m^2c^2}{\hbar^2}\phi = \mu^2\phi$$

Therefore, we can derive, $\mu = \frac{mc}{\hbar}$.

The scalar and vector fields

The scalar field is defined by the Lorentz transformation. The scalar Klein-Gordon equation is:

$$(\Box + \mu^2)\phi(x) = 0$$

The vector field is defined by the Lorentz behavior, $\phi'_\mu(x') = a^\nu_\mu \phi_\nu(x)$. The vector Klein-Gordon equation is:

$$(\Box + \mu^2)\phi_\mu(x) = 0$$

which is based on the generalized Lorentz condition:

$$\partial^\mu \phi_\mu(x) = 0$$

Let

$$G_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$$

Then, we have Proca equation:

$$\partial^\mu G_{\mu\nu} + m^2 \phi_\nu(x) = 0$$

When $m = 0$ in above, it expresses the electromagnetic field. Here is a summary

Names of equations	Spin
Klein-Goldon equation	0
Dirac equation	$\frac{1}{2}$
Proca equation	1

References

[1] J. J. Sakurai, *Advanced Quantum Mechanics*. Addison-Wesley,1967