The scattering problem for quantum mechanics III

Problem:
Find the phase shift in the field, $V = A/r^2$; then, find the scattering cross-section at small angles.

Solution:
The radial function satisfies the equation:

$$\chi_l'' + \left[k^2 - \frac{l(l+1)}{r^2} - \frac{2\mu}{\hbar^2}\frac{A}{r^2}\right]\chi = 0$$

The boundary conditions are, $\chi(0) = 0$ and $\chi(r \to \infty) = \text{finite}$. The wave function satisfying the conditions will be

$$\chi_l = \sqrt{r} J_\lambda(kr)$$

The asymptotic behavior of $J_\lambda(kr)$ indicates the phase shift:

$$\delta_l = -\frac{\pi}{2}\left(\lambda - l - \frac{1}{2}\right)$$

where

$$\lambda = \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu A}{\hbar^2}}$$

Since the phase shift is independent of *k*; thus, the scattering amplitude is

$$f(\theta, k) = \frac{1}{k} f_0(\theta)$$

where $f_0(\theta)$ is independent of the energy of the scattered particles. The scattering cross-section is:

$$d\sigma = \frac{1}{k^2}\left|f_0(\theta)^2\right| d\Omega$$

This is inversely proportional to the energy and characterized by a general angular distribution. The expression of the amplitude with the series expansion is:

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\left[\exp(2i\delta) - 1\right]$$

This indicates when $\theta \to 0$, it will diverges. The large numbers of *l* contributes to the amplitude with small values of θ. For a large l, the phase shift is approximated as

$$-\delta_l \approx -\frac{\pi\mu A}{(2l+1)\hbar^2} << 1$$

Thus, the amplitude becomes

$$f(\theta) \approx \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\delta_l$$

$$\approx -\frac{\pi\mu}{\hbar^2}\frac{A}{k}\sum_{l=0}^{\infty}P_l(\cos\theta)$$

$$\approx -\frac{\pi\mu}{\hbar^2}\frac{A}{k}\frac{1}{2\sin\frac{1}{2}\theta}$$

Then, we have

$$d\sigma = \frac{\pi^3\mu A^2}{2\hbar^2 E}\cot\frac{1}{2}\theta\, d\theta$$